# CHAPTER 5

# Reactor Dynamics

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***Summary**:*

*This chapter addresses the time-dependent behaviour of nuclear reactors. This chapter is concerned with short- and medium-time phenomena. Long-time phenomena are studied in the context of fuel and fuel cycles and are presented in Chapters 6 and 7. The chapter starts with an introduction to delayed neutrons because they play an important role in reactor dynamics. Subsequent sections present the time-dependent neutron-balance equation, starting with "point" kinetics and progressing to detailed space-energy-time methods. Effects of Xe and Sm "poisoning" are studied in Section 7, and feedback effects are presented in Section 8. Section 9 is identifies and presents the specific features of CANDU reactors as they relate to kinetics and dynamics.*

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# 1 Introduction

## 1.1 Overview

The previous chapter was devoted to predicting the neutron flux in a nuclear reactor under special steady-state conditions in which all parameters, including the neutron flux, are constant over time. During steady-state operation, the rate of neutron production must equal the rate of neutron loss. To ensure this equality, the effective multiplication factor, $k_{\text{eff}}$, was introduced as a divisor of the neutron production rate. This chapter addresses the time-dependent behaviour of nuclear reactors. In the general time-dependent case, the neutron production rate is not necessarily equal to the neutron loss rate, and consequently an overall increase or decrease in the neutron population will occur over time.

The study of the time-dependence of the neutron flux for postulated changes in the macroscopic cross sections is usually referred to as *reactor kinetics*, or reactor kinetics without feedback. If the macroscopic cross sections are allowed to depend in turn on the neutron flux level, the resulting analysis is called *reactor dynamics* or reactor kinetics with feedback.

Time-dependent phenomena are also classified by the time scale over which they occur:

- *Short-time phenomena* are phenomena in which significant changes in reactor properties occur over times shorter than a few seconds. Most accidents fall into this category.
- *Medium-time phenomena* are phenomena in which significant changes in reactor properties occur over the course of several hours to a few days. Xe poisoning is an example of a medium-time phenomenon.
- *Long-time phenomena* are phenomena in which significant changes in reactor properties occur over months or even years. An example of a long-time phenomenon is the change in fuel composition as a result of burn-up.

This chapter is concerned with short- and medium-time phenomena. Long-time phenomena are studied in the context of fuel and fuel cycles and are presented in Chapters 6 and 7. The chapter starts with an introduction to delayed neutrons because they play an important role in reactor dynamics. Subsequent sections present the time-dependent neutron-balance equation, starting with "point" kinetics and progressing to detailed space-energy-time methods. Effects of Xe and Sm "poisoning" are studied in Section 7, and feedback effects are presented in Section 8. Section 9 identifies and presents the specific features of CANDU reactors as they relate to kinetics and dynamics.

## 1.2 Learning Outcomes

The goal of this chapter is for the student to understand:

- The production of prompt and delayed neutrons through fission.
- The simple derivation of the point-kinetics equations.
- The significance of kinetics parameters such as generation time, lifetime, reactivity, and effective delayed-neutron fraction.
- Features of point-kinetics equations and how they relate to reactor behaviour (e.g., reactor period).
- Approximations involved in different kinetics models based on flux factorization.

- First-order perturbation theory.
- Fission-product poisoning.
- Reactivity coefficients and feedback.
- CANDU-specific features (long generation time, photo-neutrons, CANDU start-up, etc.).
- Numerical methods for reactor kinetics.

# 2 Delayed Neutrons

## 2.1 Production of Prompt and Delayed Neutrons: Precursors and Emitters

Binary fission of a target nucleus ${}^{A_X}_{Z_X}X$ occurs through the formation of a compound nucleus ${}^{A_X+1}_{Z_X}X$ which subsequently decays very rapidly (promptly) into two (hence the name "binary") fission products $A_m$ and $B_m$, accompanied by the emission of (prompt) gamma photons and (prompt) neutrons:

$$ {}^{1}_{0}n + {}^{A_X}_{Z_X}X \rightarrow {}^{A_X+1}_{Z_X}X \ , \tag{1} $$

(2)

The exact species of fission products $A_m$ and $B_m$, as well as the exact number of prompt neutrons emitted, $\nu_{pm}$, and the number and energy of emitted gamma photons depend on the mode $m$ according to which the compound nucleus decays. Several hundred decay modes are possible, each characterized by its probability of occurrence $p_m$. On average, $\nu_p$ prompt neutrons are emitted per fission. The average number of prompt neutrons can be expressed as:

$$ \nu_p = \sum_m p_m \nu_{pm} \ . \tag{3} $$

Obviously, although the number of prompt neutrons emitted in each decay mode, $\nu_{pm}$, is a positive integer (1, 2, 3…), the average number of neutrons emitted per fission, $\nu_p$, is a fractional number. $\nu_{pm}$ as well as $\nu_p$ depend on the target nucleus species and on the energy of the incident neutron.

The initial fission products $A_m$ and $B_m$ can be stable or can further decay in several possible modes, as shown below for $A_m$ (a similar scheme exists for $B_m$):

$$ A_m \begin{cases} \xrightarrow{\lambda} & A_{m1} + \gamma & \text{(mode 1)} \\ \xrightarrow{\lambda} & A_{m2} + {}^{0}_{-1}\beta_{HE} & \text{(mode 2)} \\ \xrightarrow{\lambda} & A'_{m3} + {}^{0}_{-1}\beta_{LE} & \text{(mode 3)} \\ & \downarrow \text{(fast)} & \\ & A_{m3} + n \ \text{(delayed neutron)} & \end{cases} . $$

Fission products $A_m$ that decay according to mode 3, by emitting a low-energy beta particle, are

called *precursors*, and the intermediate nuclides $A'_{m3}$ are called *emitters*. Emitters are daughters of precursors that are born in a highly excited state. Because their excitation energy is higher than the separation energy for one neutron, emitters can de-excite by promptly emitting a neutron. The delay in the appearance of the neutron is not caused by its emission, but rather by the delay in the beta decay of the precursor. If a high-energy beta particle rather than a low-energy one is emitted, the excitation energy of the daughter nuclide is not high enough for it to emit a neutron, and hence decay mode 2 does not result in emission of delayed neutrons.

## 2.2 Prompt, Delayed, and Total Neutron Yields

Although one cannot predict in advance which fission products will act as precursors, one can predict how many precursors on average will be produced per fission. This number is also equal to the number of delayed neutrons ultimately emitted and is called the *delayed-neutron yield*, $\nu_d$. For incident neutron energies below 4 MeV, the delayed-neutron yield is essentially independent of the incident-neutron energy. If delayed neutrons are to be represented explicitly, the fission reaction can be written generically as:

$$n + X \rightarrow A + B + \nu_p n_p + \nu_d n_d + \beta + \gamma \ . \tag{4}$$

The *total neutron yield* is defined as the sum of the prompt and delayed neutron yields:

$$\nu = \nu_d + \nu_p \ . \tag{5}$$

The delayed-neutron fraction is defined as the ratio between the delayed-neutron yield and the total neutron yield:

$$\beta = \frac{\nu_d}{\nu} \ . \tag{6}$$

For neutron energies typical of those found in a nuclear reactor, most of the energy dependence of the delayed-neutron fraction is due to the energy dependence of the prompt-neutron yield and not to that of the delayed-neutron yield. This is the case because the latter is essentially independent of energy for incident neutrons with energies below 4 MeV.

## 2.3 Delayed-Neutron Groups

Precursors can be grouped according to their half-lives. Such groups are called *precursor groups* or *delayed-neutron groups*. It is customary to use six delayed-neutron groups, but fewer or more groups can be used. For analysis of timeframes of the order of 5 seconds, six delayed-neutron groups generally provide sufficient accuracy; for longer timeframes, a greater number of groups might be needed. Partial delayed-neutron yields $\nu_{dk}$ are defined for each precursor group *k*. The partial delayed-neutron group yield represents the average number of precursors belonging to group *k* that are produced per fission. Correspondingly, partial delayed-neutron fractions can be defined as:

$$\beta_k = \frac{\nu_{dk}}{\nu} \ . \tag{7}$$

Obviously,

$$\sum_{k=1}^{k_{\max}} \beta_k = \beta \,. \tag{8}$$

Values of delayed-group constants for $^{235}$U are shown in Table 1, which uses data from [Rose1991].

**Table 1 Delayed-neutron data for thermal fission in $^{235}$U ([Rose1991])**

| Group | Decay Constant, $\lambda_k$ ($s^{-1}$) | Delayed Yield, $\nu_{dk}$ (n/fiss.) | Delayed Fraction, $\beta_k$ |
|---|---|---|---|
| 1 | 0.01334 | 0.000585 | 0.000240 |
| 2 | 0.03274 | 0.003018 | 0.001238 |
| 3 | 0.1208 | 0.002881 | 0.001182 |
| 4 | 0.3028 | 0.006459 | 0.002651 |
| 5 | 0.8495 | 0.002648 | 0.001087 |
| 6 | 2.853 | 0.001109 | 0.000455 |
| Total | - | 0.016700 | 0.006854 |

# 3 Simple Point-Kinetics Equation (Homogeneous Reactor)

This section presents the derivation of the point-kinetics equations starting from the time-dependent one-energy-group diffusion equation for the simple case of a homogeneous reactor and assuming all fission neutrons to be prompt.

## 3.1 Neutron-Balance Equation without Delayed Neutrons

The time-dependent one-energy-group diffusion equation for a homogeneous reactor without delayed neutrons can be written as:

$$\frac{\partial n(\vec{r},t)}{\partial t} = \nu\Sigma_f \Phi(\vec{r},t) + D\nabla^2\Phi(\vec{r},t) - \Sigma_a\Phi(\vec{r},t) \,, \tag{9}$$

where *n* represents the neutron density, $\nu\Sigma_f$ is the macroscopic production cross section, $\Sigma_a$ is the macroscopic neutron-absorption cross section, $\Phi$ is the neutron flux, and *D* is the diffusion coefficient. Equation (9) expresses the fact that the rate of change in neutron density at any given point is the difference between the fission source, expressed by the term $\nu\Sigma_f\Phi(\vec{r},t)$, and the two sinks: the absorption rate, expressed by the term $\Sigma_a\Phi(\vec{r},t)$, and the leakage rate, expressed by the term $-D\nabla^2\Phi(\vec{r},t)$. If the source is exactly equal to the sum of the sinks, the reactor is critical, the time dependence is eliminated, and the static balance equation results:

$$0 = \nu\Sigma_f \Phi_s(\vec{r}) + D\nabla^2\Phi_s(\vec{r}) - \Sigma_a\Phi_s(\vec{r}) \,, \tag{10}$$

which is more customarily written as:

$$-D\nabla^2\Phi_s(\vec{r}) + \Sigma_a\Phi_s(\vec{r}) = \nu\Sigma_f\Phi_s(\vec{r}) \,. \tag{11}$$

To maintain the static form of the diffusion equation even when the fission source does not exactly equal the sum of the sinks, the practice in reactor statics is to divide the fission source artificially by the effective multiplication constant, $k_{eff}$, which results in the static balance equation for a non-critical reactor:

$$-D\nabla^2\Phi_s(\vec{r})+\Sigma_a\Phi_s(\vec{r})=\frac{1}{k_{eff}}\nu\Sigma_f\Phi_s(\vec{r}) \quad . \tag{12}$$

Using the expression for the geometric buckling:

$$B_g^2=\frac{\frac{\nu\Sigma_f}{k_{eff}}-\Sigma_a}{D} \quad , \tag{13}$$

Eq. (12) becomes:

$$\nabla^2\Phi_s(\vec{r})+B_g^2\Phi_s(\vec{r})=0 \quad . \tag{14}$$

Note that the geometrical buckling is determined solely by the reactor shape and size and is independent of the production or absorption macroscopic cross sections. It follows that changes in the macroscopic cross sections do not influence buckling; they influence only the effective multiplication constant, which can be calculated as:

$$k_{eff}=\frac{\nu\Sigma_f}{\Sigma_a+DB_g^2} \quad . \tag{15}$$

Because the value of geometrical buckling is independent of the macroscopic cross section, the shape of the static flux is independent of whether or not the reactor is critical.

To progress to the derivation of the point-kinetics equation, the assumption is made that the shape of the time-dependent flux does not change with time and is equal to the shape of the static flux. In mathematical form:

$$\Phi(\vec{r},t)=T(t)\Phi_s(\vec{r}) \quad , \tag{16}$$

where *T(t)* is a function depending only on time.

It follows that the time-dependent flux $\Phi(\vec{r},t)$ satisfies Eq. (14), and hence:

$$-\nabla^2\Phi(\vec{r},t)=B_g^2\Phi(\vec{r},t) \quad . \tag{17}$$

Substituting this expression of the leakage term into the time-dependent neutron-balance equation (9) the following is obtained:

$$\frac{\partial n(\vec{r},t)}{\partial t}=\nu\Sigma_f\Phi(\vec{r},t)-DB_g^2\Phi(\vec{r},t)-\Sigma_a\Phi(\vec{r},t) \quad . \tag{18}$$

The one-group flux is the product of the neutron density and the average neutron speed, with the latter assumed to be independent of time:

$$\Phi(\vec{r},t)=n(\vec{r},t)\bar{v} \quad . \tag{19}$$

The neutron-balance equation can consequently be written as:

$$\frac{\partial n(\vec{r},t)}{\partial t} = \nu\Sigma_f \overline{\nu} n(\vec{r},t) - DB_g^2 \overline{\nu} n(\vec{r},t) - \Sigma_a \overline{\nu} n(\vec{r},t) \quad . \tag{20}$$

Integrating the local balance equation over the entire reactor volume *V*, the integral-balance equation is obtained:

$$\frac{d}{dt}\int_V n(\vec{r},t)dV = \nu\Sigma_f \overline{\nu}\int_V n(\vec{r},t)dV - DB_g^2\overline{\nu}\int_V n(\vec{r},t)dV - \Sigma_a\overline{\nu}\int_V n(\vec{r},t)dV \quad . \tag{21}$$

The volume integral of the neutron density is the total neutron population $\hat{n}(t)$, which can also be expressed as the product of the average neutron density $\overline{n}(t)$ and the reactor volume:

$$\int_V n(\vec{r},t)dV \equiv \hat{n}(t) = \overline{n}(t)V \quad . \tag{22}$$

The volume-integrated flux $\hat{\Phi}(t)$ can be defined in a similar fashion and can also be expressed as the product of the average flux $\overline{\Phi}(t)$ and the reactor volume:

$$\int_V \Phi(\vec{r},t)dV \equiv \hat{\Phi}(t) = \overline{\Phi}(t)V \quad . \tag{23}$$

It should be easy to see that the volume-integrated flux and the total neutron population satisfy a similar relationship to that satisfied by the neutron density and the neutron flux:

$$\hat{\Phi}(t) = \int_V \Phi(\vec{r},t)dV = \int_V n(\vec{r},t)\overline{\nu}dV = \hat{n}(t)\overline{\nu} \quad . \tag{24}$$

With the notations just introduced, the balance equation for the total neutron population can be written as:

$$\frac{d\hat{n}(t)}{dt} = \nu\Sigma_f \overline{\nu}\hat{n}(t) - DB_g^2\overline{\nu}\hat{n}(t) - \Sigma_a\overline{\nu}\hat{n}(t) \quad . \tag{25}$$

Equation (25) is a first-order linear differential equation, and its solution gives a full description of the time dependence of the neutron population and implicitly of the neutron flux in a homogeneous reactor without delayed neutrons. However, to highlight certain important quantities which describe the dynamic reactor behaviour, it is customary to process its right-hand side (RHS) as follows:

$$\frac{d\hat{n}(t)}{dt} = \left(\nu\Sigma_f - DB_g^2 - \Sigma_a\right)\overline{\nu}\hat{n}(t) \quad . \tag{26}$$

## 3.2 Average Neutron-Generation Time, Lifetime, and Reactivity

In this sub-section, several quantities related to neutron generation and activity are defined.

*Reactivity*

Reactivity is a measure of the relative imbalance between productions and losses. It is defined as the ratio of the difference between the production rate and the loss rate to the production rate.

$$\rho = \frac{\text{production rate - loss rate}}{\text{production rate}} = \frac{\nu\Sigma_f\hat{\Phi} - \Sigma_a\hat{\Phi} - DB_g^2\hat{\Phi}}{\nu\Sigma_f\hat{\Phi}} =$$

$$\frac{\nu\Sigma_f - \Sigma_a - DB_g^2}{\nu\Sigma_f} = 1 - \frac{\Sigma_a + DB_g^2}{\nu\Sigma_f} = 1 - \frac{\text{loss rate}}{\text{production rate}} = 1 - \frac{1}{k_{eff}} \quad . \tag{27}$$

*Average neutron-generation time*

The average neutron-generation time is the ratio between the total neutron population and the neutron production rate.

$$\Lambda = \frac{\text{neutron population}}{\text{production rate}} = \frac{\hat{n}}{\nu\Sigma_f\hat{\Phi}} = \frac{\hat{n}}{\nu\Sigma_f\hat{n}\bar{\mathrm{v}}} = \frac{1}{\nu\Sigma_f\bar{\mathrm{v}}} \quad . \tag{28}$$

The average generation time can be interpreted as the time it would take to attain the current neutron population at the current neutron-generation rate. It can also be interpreted as the average age of neutrons in the reactor.

*Average neutron lifetime*

The average neutron lifetime is the ratio between the total neutron population and the neutron loss rate:

$$\ell = \frac{\text{neutron population}}{\text{loss rate}} = \frac{\hat{n}}{\left(\Sigma_a + DB_g^2\right)\hat{\Phi}} = \frac{\hat{n}}{\left(\Sigma_a + DB_g^2\right)\hat{n}\bar{\mathrm{v}}} = \frac{1}{\left(\Sigma_a + DB_g^2\right)\bar{\mathrm{v}}} \quad . \tag{29}$$

The average neutron lifetime can be interpreted as the time it would take to lose all neutrons in the reactor at the current loss rate. It can also be interpreted as the average life expectancy of neutrons in the reactor.

The ratio of the average neutron-generation time and the average neutron lifetime equals the effective multiplication constant:

$$\frac{\ell}{\Lambda} = \frac{\dfrac{1}{\left(\Sigma_a + DB_g^2\right)\bar{\mathrm{v}}}}{\dfrac{1}{\nu\Sigma_f\bar{\mathrm{v}}}} = \frac{\nu\Sigma_f}{\Sigma_a + DB_g^2} = k_{eff} \quad . \tag{30}$$

It follows that, for a critical reactor, the neutron-generation time and the neutron lifetime are equal. It also follows that, for a supercritical reactor, the lifetime is longer than the generation time and that, for a sub-critical reactor, the lifetime is shorter than the generation time.

## 3.3 Point-Kinetics Equation without Delayed Neutrons

With the newly introduced quantities, the RHS of the neutron-balance equation (26) can be written as either:

$$\left(\nu\Sigma_f - DB_g^2 - \Sigma_a\right)\overline{\nu}\hat{n}(t) = \frac{\left(\nu\Sigma_f - DB_g^2 - \Sigma_a\right)}{\nu\Sigma_f}\left(\nu\Sigma_f\overline{\nu}\right)\hat{n}(t) = \frac{\rho}{\Lambda}\hat{n}(t) \tag{31}$$

or

$$\left(\nu\Sigma_f - DB_g^2 - \Sigma_a\right)\overline{\nu}\hat{n}(t) = \frac{\left(\nu\Sigma_f - DB_g^2 - \Sigma_a\right)}{DB_g^2 + \Sigma_a}\left(DB_g^2 + \Sigma_a\right)\overline{\nu}\hat{n}(t) = \frac{k_{eff} - 1}{\ell}\hat{n}(t) \quad . \tag{32}$$

The neutron-balance equation can therefore be written either as:

$$\frac{d\hat{n}(t)}{dt} = \frac{\rho}{\Lambda}\hat{n}(t) \tag{33}$$

or as:

$$\frac{d\hat{n}(t)}{dt} = \frac{k_{eff} - 1}{\ell}\hat{n}(t) \quad . \tag{34}$$

Equation (33), as well as Eq. (34), is referred to as the point-kinetics equation without delayed neutrons. The name *point* kinetics is used because, in this simplified formalism, the shape of the neutron flux and the neutron density distribution are ignored. The reactor is therefore reduced to a *point*, in the same way that an object is reduced to a *point mass* in simple kinematics.

Both forms of the point-kinetics equation are valid. However, because most transients are induced by changes in the absorption cross section rather than in the fission cross section, the form expressed by Eq. (33) has the mild advantage that the generation time remains constant during a transient (whereas the lifetime does not). Consequently, this text will express the neutron-balance equation using the generation time. However, the reader should be advised that other texts use the lifetime. Results obtained in the two formalisms can be shown to be equivalent.

If the reactivity and generation time remain constant during a transient, the obvious solution to the point-kinetics equation (33) is:

$$\hat{n}(t) = \hat{n}(0)e^{\frac{\rho}{\Lambda}t} \quad . \tag{35}$$

If the reactivity and generation time are not constant over time, that is, if the balance equation is written as:

$$\frac{d\hat{n}(t)}{dt} = \frac{\rho(t)}{\Lambda(t)}\hat{n}(t) \quad , \tag{36}$$

the solution becomes slightly more involved and usually proceeds either by using the Laplace transform or by time discretization.

Before advancing to accounting for delayed neutrons, one last remark will be made regarding the relationship between the neutron population and reactor power. Because the reactor power is the product of total fission rate and energy liberated per fission, it can be expressed as:

$$P(t) = E_{fiss}\Sigma_f \hat{\Phi}(t) = E_{fiss}\Sigma_f \hat{n}(t)\overline{v} \quad . \tag{37}$$

It can therefore be seen that the power has the same time dependence as the total neutron population.

## 3.4 Neutron-Balance Equation with Delayed Neutrons

In the previous section, it was assumed that all neutrons resulting from fission were prompt. This section takes a closer look and accounts for the fact that some of the neutrons are in fact delayed neutrons resulting from emitter decay.

### 3.4.1 Case of one delayed-neutron group

As explained in Section 2.1, delayed neutrons are emitted by emitters, which are daughter nuclides of precursors coming out of fission. Because the neutron-emission process occurs promptly after the creation of an emitter, the rate of delayed-neutron emission equals the rate of emitter creation and equals the rate of precursor decay. It was explained in Section 2.3 that precursors can be grouped by their half-life (or decay constant) into several (most commonly six) groups. However, as a first approximation, it can be assumed that all precursors can be lumped into a single group with an average decay constant $\lambda$. If the total concentration of precursors is denoted by $C(\vec{r},t)$, the total number of precursors in the core, $\hat{C}(t)$, is simply the volume integral of the precursor concentration and equals the product of the average precursor concentration $\overline{C}(t)$ and the reactor volume:

$$\int_V C(\vec{r},t)dV \equiv \hat{C}(t) = \overline{C}(t)V \quad . \tag{38}$$

It follows that the delayed-neutron production rate $S_d(\vec{r},t)$, which equals the precursor decay rate, is:

$$S_d(\vec{r},t) = \lambda C(\vec{r},t) \quad . \tag{39}$$

The corresponding volume-integrated quantities satisfy a similar relationship:

$$\hat{S}_d(t) = \lambda \hat{C}(t) \quad . \tag{40}$$

The core-integrated neutron-balance equation now must account explicitly for both the prompt neutron source, $\nu_p\Sigma_f\hat{\Phi}(t)$, and the delayed-neutron source:

$$\frac{d\hat{n}(t)}{dt} = \nu_p\Sigma_f\overline{v}\hat{n}(t) + \hat{S}_d(t) - DB_g^2\overline{v}\hat{n}(t) - \Sigma_a\overline{v}\hat{n}(t) =$$
$$\nu_p\Sigma_f\overline{v}\hat{n}(t) + \lambda\hat{C}(t) - DB_g^2\overline{v}\hat{n}(t) - \Sigma_a\overline{v}\hat{n}(t) \quad . \tag{41}$$

Of course, to be able to evaluate the delayed-neutron source, a balance equation for the precursors must be written as well. Precursors are produced from fission and are lost as a result of decay. It follows that the precursor-balance equation can be written as:

$$\frac{d\hat{C}(t)}{dt} = \nu_d \Sigma_f \hat{\Phi}(t) - \lambda \hat{C}(t) = \nu_d \Sigma_f \overline{\nu}\hat{n}(t) - \lambda \hat{C}(t) \quad . \tag{42}$$

The system of equations (41) and (42) completely describes the time dependence of the neutron and precursor populations. Just as in the case without delayed neutrons, they will be processed to highlight neutron-generation time and reactivity. Because reactivity is based on the total neutron yield rather than the prompt-neutron yield, the prompt-neutron source is expressed as the difference between the total neutron source and the delayed-neutron source:

$$\frac{d\hat{n}(t)}{dt} = \nu\Sigma_f \overline{\nu}\hat{n}(t) - \nu_d \Sigma_f \overline{\nu}\hat{n}(t) + \lambda \hat{C}(t) - DB_g^2 \overline{\nu}\hat{n}(t) - \Sigma_a \overline{\nu}\hat{n}(t) = \left(\nu\Sigma_f - DB_g^2 - \Sigma_a\right)\overline{\nu}\hat{n}(t) - \nu_d \Sigma_f \overline{\nu}\hat{n}(t) + \lambda \hat{C}(t) \quad . \tag{43}$$

The RHS is subsequently processed in a similar way to the no-delayed-neutron case:

$$\left(\nu\Sigma_f - DB_g^2 - \Sigma_a\right)\overline{\nu}\hat{n}(t) - \nu_d \Sigma_f \overline{\nu}\hat{n}(t) + \lambda \hat{C}(t) = \left[\frac{\left(\nu\Sigma_f - DB_g^2 - \Sigma_a\right)}{\nu\Sigma_f} - \frac{\nu_d \Sigma_f}{\nu\Sigma_f}\right]\left(\nu\Sigma_f \overline{\nu}\right)\hat{n}(t) + \lambda \hat{C}(t) = \frac{\rho - \beta}{\Lambda}\hat{n}(t) + \lambda \hat{C}(t) \quad . \tag{44}$$

The neutron-balance equation can hence be written as:

$$\frac{d\hat{n}(t)}{dt} = \frac{\rho - \beta}{\Lambda}\hat{n}(t) + \lambda \hat{C}(t) \quad . \tag{45}$$

The RHS of the precursor-balance equation can be similarly processed:

$$\nu_d \Sigma_f \overline{\nu}\hat{n}(t) - \lambda \hat{C}(t) = \frac{\nu_d \Sigma_f}{\nu\Sigma_f}\nu\Sigma_f \overline{\nu}\hat{n}(t) - \lambda \hat{C}(t) = \frac{\beta}{\Lambda}\hat{n}(t) - \lambda \hat{C}(t) \quad , \tag{46}$$

leading to the following form of the precursor-balance equation:

$$\frac{d\hat{C}(t)}{dt} = \frac{\beta}{\Lambda}\hat{n}(t) - \lambda \hat{C}(t) \quad . \tag{47}$$

Combining Eqs. (45) and (47), the system of point-kinetics equations for the case of one delayed-neutron group is obtained:

$$\frac{d\hat{n}(t)}{dt} = \frac{\rho - \beta}{\Lambda}\hat{n}(t) + \lambda \hat{C}(t)$$
$$\frac{d\hat{C}(t)}{dt} = \frac{\beta}{\Lambda}\hat{n}(t) - \lambda \hat{C}(t) \quad . \tag{48}$$

### 3.4.2 Case of several delayed-neutron groups

If the assumption that all precursors can be lumped into one single group is dropped and several precursor groups are considered, each with its own decay constant $\lambda_k$, then the delayed-neutron source is the sum of the delayed-neutron sources in all groups:

$$\hat{S}_d(t) = \sum_{k=1}^{k_{\max}} \lambda_k \hat{C}_k(t) \quad , \tag{49}$$

where $\hat{C}_k(t)$ represent the total population of precursors in group *k*.

The neutron-balance equation then becomes:

$$\frac{d\hat{n}(t)}{dt} = \nu_p \Sigma_f \overline{v}\hat{n}(t) + \hat{S}_d(t) - DB_g^2 \overline{v}\hat{n}(t) - \Sigma_a \overline{v}\hat{n}(t) =$$

$$\nu_p \Sigma_f \overline{v}\hat{n}(t) + \sum_{k=1}^{k_{\max}} \lambda_k \hat{C}_k(t) - DB_g^2 \overline{v}\hat{n}(t) - \Sigma_a \overline{v}\hat{n}(t) \quad . \tag{50}$$

Processing similar to the one-delayed-group case yields:

$$\frac{d\hat{n}(t)}{dt} = \frac{\rho - \beta}{\Lambda} \hat{n}(t) + \sum_{k=1}^{k_{\max}} \lambda_k \hat{C}_k(t) \quad . \tag{51}$$

Obviously, $k_{max}$ precursor-balance equations must now be written, one for each delayed group *k*:

$$\frac{d\hat{C}_k(t)}{dt} = \nu_{dk} \Sigma_f \overline{v}\hat{n}(t) - \lambda_k \hat{C}_k(t) \quad (k = 1 \ldots k_{\max}) \quad . \tag{52}$$

Processing the RHS of Eq. (52) as in the one-delayed-group case yields:

$$\nu_{dk} \Sigma_f \overline{v}\hat{n}(t) - \lambda_k \hat{C}_k(t) = \frac{\nu_{dk} \cancel{\Sigma_f}}{\nu \cancel{\Sigma_f}} \nu \Sigma_f \overline{v}\hat{n}(t) - \lambda_k \hat{C}_k(t) = \frac{\beta_k}{\Lambda} \hat{n}(t) - \lambda_k \hat{C}_k(t) \quad . \tag{53}$$

Finally, a system of $k_{max}+1$ differential equations is obtained:

$$\frac{d\hat{n}(t)}{dt} = \frac{\rho - \beta}{\Lambda} \hat{n}(t) + \sum_{k=1}^{k_{\max}} \lambda_k \hat{C}_k(t)$$

$$\frac{d\hat{C}_k(t)}{dt} = \frac{\beta_k}{\Lambda} \hat{n}(t) - \lambda_k \hat{C}_k(t) \quad (k = 1 \ldots k_{\max}) \quad , \tag{54}$$

representing the point-kinetics equations for the case with multiple delayed-neutron groups.

# 4 Solutions of the Point-Kinetics Equations

Following the derivation of the point-kinetics equations in the previous section, this section deals with solving the point-kinetics equations for several particular cases. The first case involves a steady-state (no time dependence) sub-critical nuclear reactor with an external neutron source constant over time. An external neutron source is a source which is independent of the neutron flux. The second case involves a single delayed-neutron group, for which an

analytical solution can be easily found if the reactivity and generation time are constant. Finally, the general outline of the solution method for the case with several delayed-neutron groups is presented.

## 4.1 Stationary Solution: Source Multiplication Formula

Possibly the simplest application of the point-kinetics equations involves a steady-state sub-critical rector with an external neutron source, that is, a source that is independent of the neutron flux. The strength of the external source is assumed constant over time. If the total strength of the source is $\hat{S}$ (n/s), the neutron-balance equation needs to be modified to include this additional source of neutrons. The precursor-balance equations remain unchanged by the presence of the external neutron source. Because a steady-state solution is sought, the time derivatives on the left-hand side (LHS) of the point-kinetics equations vanish. The steady-state point-kinetics equations in the presence of an external source can therefore be written as:

$$\begin{aligned} 0 &= \frac{\rho-\beta}{\Lambda}\hat{n}+\sum_{k=1}^{k_{\max}}\lambda_k\hat{C}_k+\hat{S} \\ 0 &= \frac{\beta_k}{\Lambda}\hat{n}-\lambda_k\hat{C}_k \quad (k=1...k_{\max}) \end{aligned} . \tag{55}$$

Equation (55) is a system of linear algebraic equations where the unknowns are the neutron and precursor populations. This can be easily seen by rearranging as follows:

$$\begin{aligned} \frac{\rho-\beta}{\Lambda}\hat{n}+\sum_{k=1}^{k_{\max}}\lambda_k\hat{C}_k &= -\hat{S} \\ \frac{\beta_k}{\Lambda}\hat{n}-\lambda_k\hat{C}_k &= 0 \quad (k=1...k_{\max}) \end{aligned} . \tag{56}$$

The system can be easily solved by substitution, by formally solving for the precursor populations in the precursor-balance equations:

$$\hat{C}_k = \frac{\beta_k}{\lambda_k\Lambda}\hat{n} \quad (k=1...k_{\max}) \tag{57}$$

and substituting the resulting expression into the neutron-balance equation to obtain:

$$\frac{\rho-\beta}{\Lambda}\hat{n}+\sum_{k=1}^{k_{\max}}\frac{\beta_k}{\Lambda}\hat{n} = -\hat{S} . \tag{58}$$

Noting that the sum of the partial delayed-neutron fractions equals the total delayed-neutron fraction, as expressed by Eq. (58), the neutron-balance equation can be processed to yield:

$$\frac{\rho}{\Lambda}\hat{n}-\frac{\beta}{\Lambda}\hat{n}+\frac{1}{\Lambda}\left(\sum_{k=1}^{k_{\max}}\beta_k\right)\hat{n} = \frac{\rho}{\Lambda}\hat{n}-\frac{\beta}{\Lambda}\hat{n}+\frac{\beta}{\Lambda}\hat{n} = \frac{\rho}{\Lambda}\hat{n} = -\hat{S} . \tag{59}$$

The neutron population is hence equal to:

$$\hat{n} = -\hat{S}\frac{\Lambda}{\rho} . \tag{60}$$

Note that the reactivity is negative, and therefore the neutron population is positive. Equation (60) is called the *source multiplication formula*. It shows that the neutron population can be obtained by multiplying the external source strength by the inverse of the reactivity, hence the name. The closer the reactor is to criticality, the larger the source multiplication and hence the neutron population. Substituting Eq. (60) into Eq. (57), the individual precursor concentrations become:

$$\hat{C}_k = \frac{\hat{S}}{-\rho}\frac{\beta_k}{\lambda_k} \quad (k = 1...k_{\max}) \quad . \tag{61}$$

The source multiplication formula finds applications in describing the approach to critical during reactor start-up and in measuring reactivity-device worth.

## 4.2 Kinetics with One Group of Delayed Neutrons

Another instance in which a simple analytical solution to the point-kinetics equations can be developed is the case of a single delayed-neutron group. This sub-section develops and analyzes the properties of such a solution. The starting point is the system of differential equations representing the neutron-balance and precursor-balance equations:

$$\begin{aligned} \frac{d\hat{n}(t)}{dt} &= \frac{\rho-\beta}{\Lambda}\hat{n}(t) + \lambda\hat{C}(t) \\ \frac{d\hat{C}(t)}{dt} &= \frac{\beta}{\Lambda}\hat{n}(t) - \lambda\hat{C}(t) \end{aligned} \quad . \tag{62}$$

For the case where all kinetics parameters are constant over time, this is a system of linear differential equations with constant coefficients, which can be rewritten in matrix form as:

$$\frac{d}{dt}\begin{bmatrix} \hat{n}(t) \\ \hat{C}(t) \end{bmatrix} = \begin{bmatrix} \frac{\rho-\beta}{\Lambda} & \lambda \\ \frac{\beta}{\Lambda} & -\lambda \end{bmatrix}\begin{bmatrix} \hat{n}(t) \\ \hat{C}(t) \end{bmatrix} \quad . \tag{63}$$

According to the general theory of systems of ordinary differential equations, the first step in solving Eq. (63) is to find two fundamental solutions of the type:

$$\begin{bmatrix} n \\ C \end{bmatrix} e^{\omega t} \quad . \tag{64}$$

The general solution can subsequently be expressed as a linear combination of the two fundamental solutions:

$$\begin{bmatrix} \hat{n}(t) \\ \hat{C}(t) \end{bmatrix} = a_0\begin{bmatrix} n_0 \\ C_0 \end{bmatrix} e^{\omega_0 t} + a_1\begin{bmatrix} n_1 \\ C_1 \end{bmatrix} e^{\omega_1 t} \quad . \tag{65}$$

Coefficients $a_0$ and $a_1$ are found by applying the initial conditions.

To find the fundamental solutions, expression (64) is substituted into Eq. (63) to obtain:

$$\frac{d}{dt}\left(\begin{bmatrix} n \\ C \end{bmatrix} e^{\omega t}\right) = \begin{bmatrix} \frac{\rho-\beta}{\Lambda} & \lambda \\ \frac{\beta}{\Lambda} & -\lambda \end{bmatrix} \begin{bmatrix} n \\ C \end{bmatrix} e^{\omega t} \quad , \tag{66}$$

and subsequently:

$$\omega \begin{bmatrix} n \\ C \end{bmatrix} e^{\omega t} = \begin{bmatrix} \frac{\rho-\beta}{\Lambda} & \lambda \\ \frac{\beta}{\Lambda} & -\lambda \end{bmatrix} \begin{bmatrix} n \\ C \end{bmatrix} e^{\omega t} \quad . \tag{67}$$

Dividing both sides by the exponential term and rearranging the terms, the following homogeneous linear system is obtained, called the *characteristic* system:

$$\begin{bmatrix} \frac{\rho-\beta}{\Lambda} - \omega & \lambda \\ \frac{\beta}{\Lambda} & -\lambda - \omega \end{bmatrix} \begin{bmatrix} n \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad . \tag{68}$$

This represents an eigenvalue-eigenvector problem, for which a solution is presented below. First, the system is rearranged so that the unknowns are each isolated on one side, and the system is rewritten as a regular system of two equations:

$$\begin{aligned} \left(\frac{\rho-\beta}{\Lambda} - \omega\right) n &= -\lambda C \\ \frac{\beta}{\Lambda} n &= (\lambda + \omega) C \end{aligned} \quad . \tag{69}$$

Dividing the two equations side by side, an equation for the eigenvalues $\omega_k$ is obtained:

$$\frac{\left(\frac{\rho-\beta}{\Lambda} - \omega\right)}{\frac{\beta}{\Lambda}} = -\frac{\lambda}{(\lambda + \omega)} \quad . \tag{70}$$

This is a quadratic equation in $\omega$, as can easily be seen after rearranging it to:

$$\omega^2 - \omega\left(\frac{\rho-\beta}{\Lambda} - \lambda\right) - \lambda\frac{\rho}{\Lambda} = 0 \quad . \tag{71}$$

The two solutions to this quadratic equation are simply:

$$\omega_{0,1} = \frac{\left(\frac{\rho-\beta}{\Lambda}-\lambda\right) \pm \sqrt{\left(\frac{\rho-\beta}{\Lambda}-\lambda\right)^2 + 4\lambda\frac{\rho}{\Lambda}}}{2} \quad . \tag{72}$$

Once the eigenvalues are known, either the first or the second of equations (69) can be used to find the relationship between $n$ and $C$. In doing so, care must be taken that the right eigenvalue (correct subscript) is used for the right *n* - *C* combination. Note that only the ratio of *n* and *C* can be determined. It follows that either *n* or *C* can have an arbitrary value, which is usually chosen to be unity. For example, if the second equation (69) is used, and if $n_0$ and $n_1$ are chosen to be unity, the two fundamental solutions are:

$$\begin{bmatrix} 1 \\ \dfrac{\beta}{\Lambda(\lambda+\omega_0)} \end{bmatrix} e^{\omega_0 t}$$

$$\begin{bmatrix} 1 \\ \dfrac{\beta}{\Lambda(\lambda+\omega_1)} \end{bmatrix} e^{\omega_1 t} \quad . \tag{73}$$

The general solution is then:

$$\begin{bmatrix} \hat{n}(t) \\ \hat{C}(t) \end{bmatrix} = a_0 \begin{bmatrix} 1 \\ \dfrac{\beta}{\Lambda(\lambda+\omega_0)} \end{bmatrix} e^{\omega_0 t} + a_1 \begin{bmatrix} 1 \\ \dfrac{\beta}{\Lambda(\lambda+\omega_1)} \end{bmatrix} e^{\omega_1 t} \quad . \tag{74}$$

## 4.3 Kinetics with Multiple Groups of Delayed Neutrons

Having solved the kinetics equations for one delayed-neutron group, it is now time to focus on the solution of the general system, with several delayed-neutron groups. The starting point is the general set of point-kinetics equations:

$$\begin{aligned} \frac{d\hat{n}(t)}{dt} &= \frac{\rho-\beta}{\Lambda}\hat{n}(t) + \sum_{k=1}^{k_{\max}} \lambda_k \hat{C}_k(t) \\ \frac{d\hat{C}_k(t)}{dt} &= \frac{\beta_k}{\Lambda}\hat{n}(t) - \lambda_k \hat{C}_k(t) \quad (k = 1,\ldots,k_{\max}) \end{aligned} \quad . \tag{75}$$

As long as the coefficients are constant, this is simply a system of first-order linear differential equations, whose general solution is a linear combination of exponential fundamental solutions of the type:

$$\begin{bmatrix} n \\ C_1 \\ \vdots \\ C_{k_{max}} \end{bmatrix} e^{\omega t} \quad . \tag{76}$$

There are $k_{max}+1$ such solutions, and the general solution can be expressed as:

$$\begin{bmatrix} \hat{n}(t) \\ \hat{C}_1(t) \\ \vdots \\ \hat{C}_{k_{max}}(t) \end{bmatrix} = \sum_{l=0}^{k_{max}} a_l \begin{bmatrix} n^l \\ C_1^l \\ \vdots \\ C_{k_{max}}^l \end{bmatrix} e^{\omega_l t} \quad . \tag{77}$$

## 4.4 Inhour Equation, Asymptotic Behaviour, and Reactor Period

By substituting the general form of the fundamental solution, Eq. (76), into the point-kinetics equations and following steps similar to those in the one-delayed-group case, the following characteristic system can be obtained:

$$\begin{aligned} \omega n &= \frac{\rho-\beta}{\Lambda} n + \sum_{k=1}^{k_{max}} \lambda_k C_k \\ \omega C_k &= \frac{\beta}{\Lambda} n - \lambda_k C_k \quad (k = 1, \ldots, k_{max}) \end{aligned} \quad . \tag{78}$$

The components $C_k$ can be expressed using the precursor equations in (78) as:

$$C_k = \frac{\beta_k}{\Lambda(\omega+\lambda_k)} n \quad (k = 1, \ldots, k_{max}) \quad . \tag{79}$$

Substituting this into the neutron-balance equation in (78) yields:

$$\omega n = \frac{\rho-\beta}{\Lambda} n + \sum_{k=1}^{k_{max}} \lambda_k \frac{\beta_k}{\Lambda(\omega+\lambda_k)} n \quad . \tag{80}$$

Note that the component $n$ can be simplified out of the above and that by rearranging terms, the following expression for reactivity is obtained:

$$\rho = \Lambda\omega + \beta - \sum_{k=1}^{k_{max}} \lambda_k \frac{\beta_k}{(\omega+\lambda_k)} \quad . \tag{81}$$

Equation (81) is known as the *Inhour equation*. Its $k_{max}+1$ solutions determine the exponents of the $k_{max}+1$ fundamental solutions. To understand the nature of those solutions, it is useful to attempt a graphical solution of the Inhour equation by plotting its RHS as a function of $\omega$ and observing its intersection points with a horizontal line at $y = \rho$. Such a plot is shown in Fig. 1 for the case of six delayed-neutron groups.

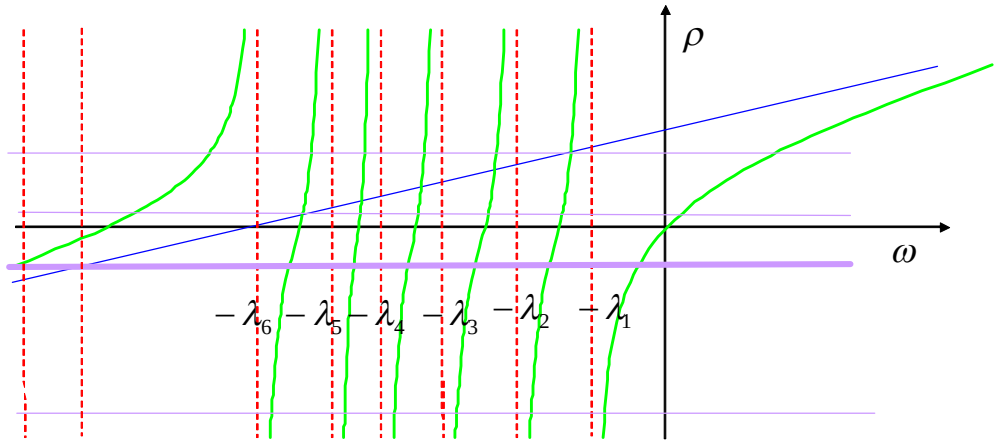


**Figure 1 Graphical representation of the Inhour equation**

For large (positive or negative) values of $\omega$, the asymptotic behaviour of the RHS can be obtained as:

$$\rho = \Lambda\omega + \beta - \sum_{k=1}^{k_{max}} \lambda_k \frac{\beta_k}{\infty + \lambda_k} = \Lambda\omega + \beta \quad . \tag{82}$$

The resulting oblique asymptote is represented by the blue line in Fig. 1, and the RHS plot (shown in bright green) approaches it at both $-\infty$ and $+\infty$. Whenever $\omega$ equals minus the decay constant for one of the precursor groups, the RHS becomes infinite, and its plot has a vertical asymptote, shown as a (red) dashed line, at that value. For $\omega = 0$, the RHS vanishes, as can be seen from Eq. (81), and hence the plot passes through the origin of the coordinate system. Three horizontal lines, corresponding to three reactivity values, are shown in violet. The two thin lines correspond to positive values, and the thick line corresponds to the negative value.

Figure 1 shows that the solutions to the Inhour equation are distributed as follows:

- $k_{max}$ - *1* solutions are located in the $k_{max}$ - *1* intervals separating the $k_{max}$ decay constants taken with negative signs, such that $-\lambda_k < \omega_{k-1} - \lambda_{k-1}$. All these solutions are negative.
- The largest solution, in an algebraic sense, is located to the right of $-\lambda_1$ and is either negative or positive, depending on the sign of the reactivity. It will be referred to as $\omega_{\max}$ or $\omega_0$.
- The smallest solution, in an algebraic sense, lies to the left of $-\lambda_{k_{\max}}$ and will be referred to as $\omega_{\min}$ or $\omega_{k_{\max}}$. It is (obviously) negative as well. Note that because the generation time is usually less than 1 ms, and often less than 0.1 ms, the slope of the oblique asymptote is very small. Consequently, $\omega_{\min}$ is very far to the left of $-\lambda_{k_{\max}}$, and hence $\omega_{k_{\max}} \ll -\lambda_{k_{\max}} < \omega_{k_{\max}-1}$. The importance of this fact will become clearer later, when the prompt-jump approximation will be discussed.

Overall, the solutions are ordered as follows:

$$\omega_{\min} \equiv \omega_{k_{\max}} \ll \omega_{k_{\max}-1} < ..... < \omega_0 \equiv \omega_{\max} . \tag{83}$$

It is worth separating out the largest exponent in the general solution described by Eq. (77) by writing:

$$\begin{bmatrix} \hat{n}(t) \\ \hat{C}_1(t) \\ \vdots \\ \hat{C}_{k_{\max}}(t) \end{bmatrix} = a_0 \begin{bmatrix} n^0 \\ C_1^0 \\ \vdots \\ C_{k_{\max}}^0 \end{bmatrix} e^{\omega_0 t} + \sum_{l=1}^{k_{\max}} a_l \begin{bmatrix} n^l \\ C_1^l \\ \vdots \\ C_{k_{\max}}^l \end{bmatrix} e^{\omega_l t} . \tag{84}$$

Furthermore, it is worth factoring out the first exponential term:

$$\begin{bmatrix} \hat{n}(t) \\ \hat{C}_1(t) \\ \vdots \\ \hat{C}_{k_{\max}}(t) \end{bmatrix} = e^{\omega_0 t} \left( a_0 \begin{bmatrix} n^0 \\ C_1^0 \\ \vdots \\ C_{k_{\max}}^0 \end{bmatrix} + \sum_{l=1}^{k_{\max}} a_l \begin{bmatrix} n^l \\ C_1^l \\ \vdots \\ C_{k_{\max}}^l \end{bmatrix} e^{(\omega_l - \omega_0)t} \right) . \tag{85}$$

Note that because $\omega_0$ is the largest solution, all exponents $(\omega_l - \omega_0)$ are negative. It follows that for large values of *t*, all exponentials of the type $e^{(\omega_l - \omega_0)t}$ nearly vanish, and hence the solution can be approximated by a single exponential term:

$$\begin{bmatrix} \hat{n}(t) \\ \hat{C}_1(t) \\ \vdots \\ \hat{C}_{k_{\max}}(t) \end{bmatrix} \cong a_0 \begin{bmatrix} n^0 \\ C_1^0 \\ \vdots \\ C_{k_{\max}}^0 \end{bmatrix} e^{\omega_0 t} . \tag{86}$$

This expression describes the *asymptotic transient behaviour*.

The inverse of $\omega_0 = \omega_{\max}$ is called the *asymptotic period*:

$$T \equiv \frac{1}{\omega_{\max}} . \tag{87}$$

With this new notation, the asymptotic behaviour can be written as:

$$\begin{bmatrix} \hat{n}(t) \\ \hat{C}_1(t) \\ \vdots \\ \hat{C}_{k_{\max}}(t) \end{bmatrix} \cong a_0 \begin{bmatrix} n^0 \\ C_1^0 \\ \vdots \\ C_{k_{\max}}^0 \end{bmatrix} e^{\frac{t}{T}} . \tag{88}$$

Before ending this sub-section, a few more comments are warranted. In particular, it is worth considering the solution to the *point-kinetics equation* (PKE) for three separate cases: negative

reactivity, zero reactivity, and positive reactivity.

Negative reactivity

In the case of negative reactivity, all exponents in the general solution are negative. It follows that over time, both the neutron population and the precursor concentrations will drop to zero. Of course, after a long time, the asymptotic behaviour applies, which has a negative exponent.

Zero reactivity

In the case of zero reactivity, $\omega_{\max}$ vanishes, and all other $\omega_l$ are negative. The general solution can be written as:

$$\begin{bmatrix} \hat{n}(t) \\ \hat{C}_1(t) \\ \vdots \\ \hat{C}_{k_{\max}}(t) \end{bmatrix} = a_0 \begin{bmatrix} n^0 \\ C_1^0 \\ \vdots \\ C_{k_{\max}}^0 \end{bmatrix} + \sum_{l=1}^{k_{\max}} a_l \begin{bmatrix} n^l \\ C_1^l \\ \vdots \\ C_{k_{\max}}^l \end{bmatrix} e^{\omega_l t} . \tag{89}$$

After a sufficiently long time, all the exponential terms die out, and the neutron and precursor populations stabilize at a constant value. Note that these populations do not need to remain constant from the beginning of the transient, but only to stabilize at a constant value.

Positive reactivity

In the case of positive reactivity, $\omega_{\max}$ is positive, and all other $\omega_l$ are negative. Hence, after sufficient time has elapsed, all but the first exponential term vanish, and the asymptotic behaviour is described by a single exponential which increases indefinitely.

## 4.5 Approximate Solution of the Point-Kinetics Equations: The Prompt Jump Approximation

It was mentioned in the preceding sub-section that the smallest (in an algebraic sense) solution of the Inhour equation is much smaller than the remaining $k_{max}$ solutions. This important property will make it possible to introduce the *prompt jump approximation*, which is the topic of this sub-section.

By inspecting the Inhour plot in Figure 1, and keeping in mind the expression of the oblique asymptote given by Eq. (82), it is easy to notice that the oblique asymptote intersects the *x*-axis at:

$$\omega_{as} = \frac{\rho - \beta}{\Lambda} . \tag{90}$$

It is also easy to see that:

$$\omega_{k_{\max}+1} < \omega_{as} . \tag{91}$$

Assuming a reactivity smaller than approximately half the delayed-neutron fraction (equal to 0.0065 according to Table 1), and assuming a generation time of approximately 0.1 ms, the

resulting value of $\omega_{as}$ is approximately -32.5 $s^{-1}$, which is much smaller than even the largest decay constant in Table 1 taken with a negative sign. That value is only -3 $s^{-1}$. This shows that the following inequality holds true:

$$\omega_{\min} \equiv \omega_{k_{\max}} < \omega_{as} << \omega_{k_{\max}-1} < ..... < \omega_0 \equiv \omega_{\max} . \tag{92}$$

The general solution of the point-kinetics equations expressed by Eq. (77) can be processed to separate out the term corresponding to $\omega_{k_{\max}-1}$:

$$\begin{bmatrix} \hat{n}(t) \\ \hat{C}_1(t) \\ \vdots \\ \hat{C}_{k_{\max}}(t) \end{bmatrix} = e^{\omega_{k_{\max}-1}t} \left( a_{k_{\max}} \begin{bmatrix} n^0 \\ C_1^0 \\ \vdots \\ C_{k_{\max}}^0 \end{bmatrix} e^{\left(\omega_{k_{\max}} - \omega_{k_{\max}-1}\right)t} + a_{k_{\max}-1} \begin{bmatrix} n^{k_{\max}-1} \\ C_1^{k_{\max}-1} \\ \vdots \\ C_{k_{\max}}^{k_{\max}-1} \end{bmatrix} + \sum_{l=0}^{k_{\max}-2} a_l \begin{bmatrix} n^l \\ C_1^l \\ \vdots \\ C_{k_{\max}}^l \end{bmatrix} e^{\left(\omega_l - \omega_{k_{\max}-1}\right)t} \right) . \tag{93}$$

According to Eq. (92), for $l \le k_{\max} - 2$, all exponents of the type $\left(\omega_l - \omega_{k_{\max}-1}\right)t$ are positive. The only negative exponent is $\left(\omega_{k_{\max}} - \omega_{k_{\max}-1}\right)t$, which is also much larger in absolute value than all other exponents. It follows that after a very short time, *t*, the first term of the RHS of Eq. (93) becomes negligible, and the solution of the point-kinetics equations can then be approximated by:

$$\begin{bmatrix} \hat{n}(t) \\ \hat{C}_1(t) \\ \vdots \\ \hat{C}_{k_{\max}}(t) \end{bmatrix} \cong e^{\omega_{k_{\max}-1}t} \left( a_{k_{\max}-1} \begin{bmatrix} n^{k_{\max}-1} \\ C_1^{k_{\max}-1} \\ \vdots \\ C_{k_{\max}}^{k_{\max}-1} \end{bmatrix} + \sum_{l=0}^{k_{\max}-2} a_l \begin{bmatrix} n^l \\ C_1^l \\ \vdots \\ C_{k_{\max}}^l \end{bmatrix} e^{\left(\omega_l - \omega_{k_{\max}-1}\right)t} \right) = \sum_{l=0}^{k_{\max}-1} a_l \begin{bmatrix} n^l \\ C_1^l \\ \vdots \\ C_{k_{\max}}^l \end{bmatrix} e^{\omega_l t} . \tag{94}$$

Concentrating on the neutron population, its expression is:

$$\hat{n}(t) \cong \sum_{l=0}^{k_{\max}-1} a_l n^l e^{\omega_l t} . \tag{95}$$

Substituting this into the neutron-balance equation of the point-kinetics system, the following is obtained:

$$\sum_{l=1}^{k_{\max}-1} \omega_l a_l n^l e^{\omega_l t} = \sum_{l=1}^{k_{\max}-1} \frac{\rho - \beta}{\Lambda} a_l n^l e^{\omega_l t} + \sum_{k=1}^{k_{\max}} \lambda_k \hat{C}_k . \tag{96}$$

Noting that the following inequality holds true:

$$\left|\omega_l\right| << \left|\frac{\rho - \beta}{\Lambda}\right| \quad l = 0, ...k_{\max} - 1 , \tag{97}$$

the LHS of Eq. (96) can be approximated to vanish, and hence the equation can be approximated by:

$$0 = \sum_{l=1}^{k_{\max}-1} \frac{\rho-\beta}{\Lambda} a_l n^l e^{\omega_l t} + \sum_{k=1}^{k_{\max}} \lambda_k \hat{C}_k \quad , \tag{98}$$

which is equivalent to:

$$0 = \frac{\rho-\beta}{\Lambda} \hat{n}(t) + \sum_{k=1}^{k_{\max}} \lambda_k \hat{C}_k(t) \quad . \tag{99}$$

By adding the precursor-balance equations, the following approximate point-kinetics equations are obtained:

$$\begin{aligned} 0 &= \frac{\rho-\beta}{\Lambda} \hat{n}(t) + \sum_{k=1}^{k_{\max}} \lambda_k \hat{C}_k(t) \\ \frac{d\hat{C}_k(t)}{dt} &= \frac{\beta_k}{\Lambda} \hat{n}(t) - \lambda_k \hat{C}_k(t) \quad (k = 1, \ldots, k_{\max}) \end{aligned} \quad . \tag{100}$$

This system of $k_{max}$ differential equations and one algebraic equation is known as the *prompt jump approximation* of the point-kinetics equations. The name comes from the fact that whenever a step reactivity change occurs, the prompt jump approximation results in a step change, a *prompt jump*, in the neutron population. To demonstrate this behaviour, let the reactivity change from $\rho_1$ to $\rho_2$ at time $t_0$. The neutron-balance equation before and after $t_0$ can be written as:

$$\begin{aligned} &\frac{\rho_1-\beta}{\Lambda} \hat{n}(t) + \sum_{k=1}^{k_{\max}} \lambda_k \hat{C}_k(t) \quad (t < t_0) \\ &\frac{\rho_2-\beta}{\Lambda} \hat{n}(t) + \sum_{k=1}^{k_{\max}} \lambda_k \hat{C}_k(t) \quad (t > t_0) \end{aligned} \quad . \tag{101}$$

The limit of the neutron population as $t$ approaches $t_0$ from the left, symbolically denoted as $\hat{n}(t_0^-)$, is found from the first equation (101) to be equal to:

$$\hat{n}(t_0^-) = \frac{\Lambda}{\rho_1-\beta} \sum_{k=1}^{k_{\max}} \lambda_k \hat{C}_k(t_0) \quad . \tag{102}$$

Similarly, the limit of the neutron population as $t$ approaches $t_0$ from the right, symbolically denoted as $\hat{n}(t_0^+)$, is found from the second equation (101) to be equal to:

$$\hat{n}(t_0^+) = \frac{\Lambda}{\rho_2-\beta} \sum_{k=1}^{k_{\max}} \lambda_k \hat{C}_k(t_0) \quad . \tag{103}$$

Taking the ratio of the preceding two equations side by side, the following is obtained:

$$\frac{\hat{n}(t_0^+)}{\hat{n}(t_0^-)} = \frac{\rho_1-\beta}{\rho_2-\beta} \quad . \tag{104}$$

There is therefore a jump $\Delta\hat{n}(t_0)$ equal to:

$$\Delta\hat{n}(t_0) = \hat{n}(t_0^+) - \hat{n}(t_0^-) = \hat{n}(t_0^-)\frac{\rho_1 - \beta}{\rho_2 - \beta} - \hat{n}(t_0^-) = \frac{\rho_1 - \rho_2}{\rho_2 - \beta}\hat{n}(t_0^-) \quad . \tag{105}$$

Of course, the actual neutron population does not display such a jump; it is continuous at $t_0$. Nonetheless, a very short time $\Delta t$ after $t_0$ (at $t_0 + \Delta t$), the approximate and exact neutron populations become almost equal. Note also that Eq. (105) is valid only if both reactivities $\rho_1$ and $\rho_2$ are less than the effective delayed-neutron fraction $\beta$.

# 5 Space-Time Kinetics using Flux Factorization

In the previous sections, the time-dependent behaviour of a reactor was studied using the simple point-kinetics model, which disregards changes in the spatial distribution of the neutron density. This section will improve on that model by presenting the general outline of space-time kinetics using flux factorization. The approach follows roughly that used in [Rozon1998] and [Ott1985]. A complete and thorough treatment of the topic of space-time kinetics is beyond the scope of this text. This section should therefore be regarded merely as a roadmap. The interested reader is encouraged to study the more detailed treatments in [Rozon1998], [Ott1985], and [Stacey1970].

## 5.1 Time-, Energy-, and Space-Dependent Multigroup Diffusion Equation

The space-time description of reactor kinetics starts with the time-, space-, and energy-dependent diffusion equation. An equivalent treatment starting from the transport equation is also possible, but using the transport equation instead of the diffusion equation does not introduce fundamentally different issues, and the mathematical treatment is somewhat more cumbersome. The time-, space- and energy-dependent neutron diffusion equation in the multigroup approximation can be written as follows:

$$\begin{aligned}
&\frac{1}{\overline{\nu}_g}\frac{\partial}{\partial t}\Phi_g(\vec{r},t) = \\
&\nabla\cdot\left[D_g(\vec{r},t)\nabla\Phi_g(\vec{r},t)\right] - \Sigma_{rg}(\vec{r},t)\Phi_g(\vec{r},t) + \sum_{g'\neq g}\Sigma_{sg'\to g}(\vec{r},t)\Phi_{g'}(\vec{r},t) + \\
&\chi_{pg}(\vec{r},t)\sum_{g'}\nu_p(\vec{r},t)\Sigma_{fg'}(\vec{r},t)\Phi_{g'}(\vec{r},t) + \sum_{k=1}^{k_{\max}}\chi_{dg}^k(\vec{r},t)\lambda_k C_k(\vec{r},t)
\end{aligned} \quad . \tag{106}$$

The accompanying precursor-balance equations are written as:

$$\frac{\partial}{\partial t}c_k(\vec{r},t) = \sum_{g'}\nu_{pk}(\vec{r},t)\Sigma_{fg'}(\vec{r},t)\Phi_{g'}(\vec{r},t) - \lambda_k C_k(\vec{r},t) \quad . \tag{107}$$

Equations (106) and (107) represent the space-time kinetics equations in their diffusion approximation. Their solution is the topic of this section.

It is advantageous for the development of the space-time kinetics formalism to introduce a set of multidimensional vectors and operators, as follows:

Flux vector

$$\mathbf{\Phi}\left(\vec{r},t\right)=\left[\Phi_g\left(\vec{r},t\right)\right] \tag{108}$$

Precursor vector

$$\boldsymbol{\xi}_k(\vec{r},t)=\left[\chi_{dg}^k C_k(\vec{r},t)\right] \tag{109}$$

Loss operator

$$\mathbf{M}\left(\vec{r},t\right)=\nabla\cdot\left[D_g(\vec{r},t)\nabla\Phi_g(\vec{r},t)\right]-\Sigma_{rg}(\vec{r},t)\Phi_g(\vec{r},t)+\sum_{g'\neq g}\Sigma_{sg'\to g}(\vec{r},t)\Phi_{g'}(\vec{r},t) \tag{110}$$

Prompt production operator

$$\mathbf{F}_p(\vec{r},t)=\chi_{pg}(\vec{r},t)\sum_{g'}\nu_p(\vec{r},t)\Sigma_{fg'}(\vec{r},t)\Phi_{g'}(\vec{r},t) \tag{111}$$

Precursor production operator for precursor group *k*

$$\mathbf{F}_{dk}(\vec{r},t)=\chi_{dg}^k(\vec{r},t)\lambda_k C_k(\vec{r},t) \tag{112}$$

Inverse-speed operator

$$\mathbf{v}^{-1}=\frac{1}{\overline{\mathrm{v}}_g}\delta_{g'g}, \tag{113}$$

where $\delta_{g'g}$ is the Kronecker delta symbol.

Using these definitions, the time-dependent multigroup diffusion equation can be written in compact form as:

$$\frac{\partial}{\partial t}\mathbf{v}^{-1}\mathbf{\Phi}\left(\vec{r},t\right)=-\mathbf{M}(\vec{r},t)\mathbf{\Phi}\left(\vec{r},t\right)+\mathbf{F}_p(\vec{r},t)\mathbf{\Phi}\left(\vec{r},t\right)+\sum_{k=1}^{k_{\max}}\lambda_k\boldsymbol{\xi}_k(\vec{r},t). \tag{114}$$

The precursor-balance equations can be written as:

$$\frac{\partial}{\partial t}\boldsymbol{\xi}_k(\vec{r},t)=\mathbf{F}_{dk}(\vec{r},t)\mathbf{\Phi}\left(\vec{r},t\right)-\lambda_k\boldsymbol{\xi}_k(\vec{r},t)\quad(k=1,\ldots,k_{\max}). \tag{115}$$

As a last definition, for two arbitrary vectors $\mathbf{\Phi}(\vec{r},t)$ and $\mathbf{\Psi}(\vec{r},t)$, the inner product is defined as:

$$\left\langle\mathbf{\Phi},\mathbf{\Psi}\right\rangle=\sum_g\int_{V_{core}}\Phi_g(\vec{r},t)\Psi_g(\vec{r},t)dV. \tag{116}$$

## 5.2 Flux Factorization

Expressing a function as a product of several (simpler) functions is known as *factorization*. It is a well-known fact from partial differential equations that trying to express the solution as a product of single-variable functions often simplifies the mathematical treatment. It is therefore reasonable to attempt a similar approach for the space-time kinetics problem. A first step in this approach is to factorize the time-, energy-, and space-dependent solution into a function

dependent only on time and a vector dependent on energy, space, and time. The function dependent only on time is called an *amplitude function*, and the vector dependent on space, energy, and time is called a *shape function*. The sought-for flux can therefore be expressed as:

$$\mathbf{\Phi}(\vec{r},t) = p(t)\mathbf{\Psi}(\vec{r},t) \quad . \tag{117}$$

Such a factorization is always possible, regardless of the definition of the function *p(t)*. In this case, the function *p(t)* is defined as follows:

$$p(t) \equiv \left\langle \mathbf{w}(\vec{r}), \mathbf{v}^{-1}\mathbf{\Phi}(\vec{r},t) \right\rangle \quad , \tag{118}$$

where $\mathbf{w}(\vec{r})$ is an arbitrary weight vector dependent only on energy and position:

$$\mathbf{w}(\vec{r}) = \left[ w_g(\vec{r}) \right] \quad . \tag{119}$$

According to its definition, *p(t)* can be interpreted as a generalized neutron population. Indeed, if the weight function were chosen to be unity, *p(t)* would be exactly equal to the neutron population.

From the definition of the flux factorization, it follows that the shape vector $\mathbf{\Psi}(\vec{r},t)$ satisfies the following normalization condition:

$$\left\langle \mathbf{w}(\vec{r}), \mathbf{v}^{-1}\mathbf{\Psi}(\vec{r},t) \right\rangle = 1 \quad . \tag{120}$$

Substituting the factorized form of the flux into the space-, energy-, and time-dependent diffusion equation, the following equations (representing respectively the neutron and precursor balance) result:

$$\frac{dp(t)}{dt}\mathbf{v}^{-1}\mathbf{\Psi}(\vec{r},t) + p(t)\frac{\partial}{\partial t}\left[\mathbf{v}^{-1}\mathbf{\Psi}(\vec{r},t)\right] = -p(t)\mathbf{M}(\vec{r},t)\mathbf{\Psi}(\vec{r},t) + p(t)\mathbf{F}_p(\vec{r},t)\mathbf{\Psi}(\vec{r},t) + \sum_{k=1}^{k_{\max}} \lambda_k \xi_k(\vec{r},t) \quad . \tag{121}$$

$$\frac{\partial}{\partial t}\xi_k(\vec{r},t) = p(t)\mathbf{F}_{dk}(\vec{r},t)\mathbf{\Psi}(\vec{r},t) - \lambda_k \xi_k(\vec{r},t) \quad (k = 1,\ldots,k_{\max}) \quad . \tag{122}$$

The precursor-balance equation can be solved formally to give:

$$\xi_k(\vec{r},t) = \xi_k(\vec{r},0)e^{-\lambda_k t} + \int_0^t e^{-\lambda_k (t-t')} p(t')\mathbf{F}_{dk}(\vec{r},t')\mathbf{\Psi}(\vec{r},t')dt' \quad . \tag{123}$$

By taking the inner product with the weight vector $\mathbf{w}(\vec{r})$ on both sides of the neutron-balance equation and the precursor-balance equation, the following is obtained:

$$\frac{dp(t)}{dt}\left\langle \mathbf{w}(\vec{r}); \mathbf{v}^{-1}\mathbf{\Psi}(\vec{r},t) \right\rangle + p(t)\frac{d}{dt}\left\langle \mathbf{w}(\vec{r}); \mathbf{v}^{-1}\mathbf{\Psi}(\vec{r},t) \right\rangle = - p(t)\left\langle \mathbf{w}(\vec{r}); \mathbf{M}(\vec{r},t)\mathbf{\Psi}(\vec{r},t) \right\rangle + p(t)\left\langle \mathbf{w}(\vec{r}); \mathbf{F}_p(\vec{r},t)\mathbf{\Psi}(\vec{r},t) \right\rangle + \sum_{k=1}^{k_{\max}} \lambda_k \left\langle \mathbf{w}(\vec{r}); \xi_k(\vec{r},t) \right\rangle \quad . \tag{124}$$

$$\frac{\partial}{\partial t}\left\langle \mathbf{w}(\vec{r});\xi_k(\vec{r},t)\right\rangle = p(t)\left\langle \mathbf{w}(\vec{r});\mathbf{F}_{dk}(\vec{r},t)\boldsymbol{\Psi}\left(\vec{r},t\right)\right\rangle - \lambda_k\left\langle \mathbf{w}(\vec{r});\xi_k(\vec{r},t)\right\rangle \quad (k=1,\ldots,k_{\max}) \tag{125}$$

Equations (124) and (125) can be processed into more elegant forms akin to the point-kinetics equations. To do this, some quantities must be defined first which will prove to be generalizations of the same quantities defined for the point-kinetics equations.

## 5.3 Effective Generation Time, Effective Delayed-Neutron Fraction, and Dynamic Reactivity

The following quantities and symbols are introduced:

Total production operator

$$\mathbf{F}(\vec{r},t) = \mathbf{F}_p(\vec{r},t) + \mathbf{F}_d(\vec{r},t) \tag{126}$$

Dynamic reactivity

$$\rho(t) = \frac{\left\langle \mathbf{w}(\vec{r}),\mathbf{F}(\vec{r},t)\boldsymbol{\Psi}(\vec{r},t)\right\rangle - \left\langle \mathbf{w}(\vec{r}),\mathbf{M}(\vec{r},t)\boldsymbol{\Psi}(\vec{r},t)\right\rangle}{\left\langle \mathbf{w}(\vec{r}),\mathbf{F}(\vec{r},t)\boldsymbol{\Psi}(\vec{r},t)\right\rangle} \tag{127}$$

Effective generation time

$$\Lambda(t) = \frac{\left\langle \mathbf{w}(\vec{r}),\mathbf{v}^{-1}\boldsymbol{\Psi}(\vec{r},t)\right\rangle}{\left\langle \mathbf{w}(\vec{r}),\mathbf{F}(\vec{r},t)\boldsymbol{\Psi}(\vec{r},t)\right\rangle} \tag{128}$$

Effective delayed-neutron fraction for delayed group *k*

$$\beta_k(t) = \frac{\left\langle \mathbf{w}(\vec{r}),\mathbf{F}_{dk}(\vec{r},t)\boldsymbol{\Psi}(\vec{r},t)\right\rangle}{\left\langle \mathbf{w}(\vec{r}),\mathbf{F}(\vec{r},t)\boldsymbol{\Psi}(\vec{r},t)\right\rangle} \tag{129}$$

Total effective delayed-neutron fraction

$$\beta(t) = \sum_{k=1}^{k_{\max}} \beta_k(t) \tag{130}$$

Group *k* (generalized) precursor population

$$\hat{C}_k(t) = \left\langle \mathbf{w}(\vec{r}),\xi_k(\vec{r},t)\right\rangle \tag{131}$$

With the newly introduced quantities, Eqs. (124) and (125) can be rewritten in the familiar form of the point-kinetics equations:

$$\dot{p}(t) = p(t)\frac{\rho(t)-\beta(t)}{\Lambda(t)} + \sum_{k=1}^{k_{\max}} \lambda_k \hat{C}_k(t)$$
$$\frac{\partial}{\partial t}\hat{C}_k(t) = p(t)\frac{\beta_k(t)}{\Lambda(t)} - \lambda_k \hat{C}_k(t) \quad (k=1,\ldots,k_{\max}) \tag{132}$$

Of course, to be able to define quantities such as the dynamic reactivity, the shape vector

$\boldsymbol{\Psi}(\vec{r},t)$ must be known or approximated at each time *t*. Different shape representations $\boldsymbol{\Psi}(\vec{r},t)$ lead to different space-time kinetics models. All flux-factorization models alternate between calculating the shape vector $\boldsymbol{\Psi}(\vec{r},t)$ and solving the point-kinetics-like equations (132) for the amplitude function and the precursor populations. The detailed energy- and space-dependent flux shape at each time *t* can subsequently be reconstructed by multiplying the amplitude function by the shape vector.

## 5.4 Improved Quasistatic Model

The *improved quasistatic* (IQS) *model* uses an exact shape $\boldsymbol{\Psi}(\vec{r},t)$. By substituting the formal solution to the precursor equations (123) into the general neutron-balance equation (121), the following equation for the shape vector is obtained:

$$\frac{dp(t)}{dt}\mathbf{v}^{-1}\boldsymbol{\Psi}\left(\vec{r},t\right)+p(t)\frac{\partial}{\partial t}\left[\mathbf{v}^{-1}\boldsymbol{\Psi}\left(\vec{r},t\right)\right]=-p(t)\mathbf{M}(\vec{r},t)\boldsymbol{\Psi}\left(\vec{r},t\right)+$$
$$p(t)\mathbf{F}_p(\vec{r},t)\boldsymbol{\Psi}\left(\vec{r},t\right)+$$
$$\sum_{k=1}^{k_{\max}}\lambda_k\left(\xi_k(0)e^{-\lambda_k t}+\int_0^t e^{-\lambda_k(t-t')}p(t')\mathbf{F}_{dk}(\vec{r},t')\boldsymbol{\Psi}(\vec{r},t')dt'\right). \qquad (133)$$

The IQS model alternates between solving the point-kinetics-like equations (132) and the shape equation (133). The corresponding IQS numerical method uses two sizes of time interval. Because the amplitude function varies much more rapidly with time than the shape vector, the time interval used to solve the point-kinetics-like equations is much smaller than that used to solve for the shape vector. Note that, other than the time discretization, the IQS model and method include no approximation. The actual shape of the weight vector $\mathbf{w}(\vec{r})$ is irrelevant.

## 5.5 Quasistatic Approximation

The quasistatic approximation is derived by neglecting the time derivative of the shape vector in the shape-vector equation (133). The resulting equation, which is solved at each time step, is:

$$\frac{dp(t)}{dt}\mathbf{v}^{-1}\boldsymbol{\Psi}\left(\vec{r},t\right)=-p(t)\mathbf{M}(\vec{r},t)\boldsymbol{\Psi}\left(\vec{r},t\right)+$$
$$p(t)\mathbf{F}_p(\vec{r},t)\boldsymbol{\Psi}\left(\vec{r},t\right)+\sum_{k=1}^{k_{\max}}\lambda_k\left(\xi_k(0)e^{-\lambda_k t}+\int_0^t e^{-\lambda_k(t-t')}p(t')\mathbf{F}_{dk}(\vec{r},t')\boldsymbol{\Psi}(\vec{r},t')dt'\right). \qquad (134)$$

The resulting shape is used to calculate the point-kinetics parameters, which are then used in the point-kinetics-like equations (132). As in the case of the IQS model, Equation (134) is solved in conjunction with the point-kinetics-like equations (132). Aside from the slightly modified shape equation, the quasistatic model differs from the IQS model in the values of its point-kinetics parameters, which are now calculated using an approximate shape vector.

## 5.6 Adiabatic Approximation

The adiabatic approximation completely does away with any time derivative in the shape equation and instead solves the static eigenvalue problem at each time *t*:

$$\mathbf{M}(\vec{r},t)\boldsymbol{\Psi}\left(\vec{r},t\right)=\frac{1}{k}\mathbf{F}\ (\vec{r},t)\boldsymbol{\Psi}(\vec{r},t) \quad . \tag{135}$$

The resulting shape is used to calculate the point-kinetics parameters, which are then used in the point-kinetics-like equations (132). As in the case of the IQS and quasistatic models, Equation (135) is solved in conjunction with the point-kinetics-like equations (132).

## 5.7 Point-Kinetics Approximation (Rigorous Derivation)

In the case of the point-kinetics model, the shape vector is determined only once at the beginning of the transient (*t=0*) by solving the static eigenvalue problem:

$$\mathbf{M}(\vec{r},0)\boldsymbol{\Psi}\left(\vec{r}\right)=\frac{1}{k}\mathbf{F}\ (\vec{r},0)\boldsymbol{\Psi}(\vec{r}) \quad . \tag{136}$$

The resulting shape is used to calculate the point-kinetics parameters, which are then used in the point-kinetics-like equations (132). Because the shape vector is not updated, only the point-kinetics-like equations (132) are solved at each time *t*. In fact, they are now the true point-kinetics equations because the shape vector remains constant over time. This discussion has shown that the point-kinetics equations can also be derived for an inhomogeneous reactor, as long as the flux is factorized into a shape vector depending only on energy and position and an amplitude function depending only on time.

# 6 Perturbation Theory

It should be obvious by now that different approximations of the shape vector lead to different values for the kinetics parameters. It is therefore of interest to determine whether certain choices of the weight vector might maintain the accuracy of the kinetics parameters even when approximate shape vectors are used. In particular, it would be interesting to obtain accurate values of the dynamic reactivity, which is the determining parameter for any transient. The issue of determining the weight function that leads to the smallest errors in reactivity when small errors exist in the shape vector is addressed by *perturbation theory*. This section will present without proof some important results of perturbation theory. The interested reader is encouraged to consult [Rozon1998], [Ott1985], and [Stacey1970] for detailed proofs and additional results.

## 6.1 Essential Results from Perturbation Theory

Perturbation theory analyzes the effect on reactivity of small changes in reactor cross sections with respect to an initial critical state, called the *reference* state. These changes are called *perturbations*, and the resulting state is called the *perturbed* state. Perturbation theory also analyzes the effect of calculating the reactivity using approximate rather than exact flux shapes. First-order perturbation theory states that the weight vector that achieves the best first-order approximation of the reactivity (e.g., for the point-kinetics equations) when using an approximate (rather than an exact) flux shape is the *adjoint function*, which is defined as the solution to the *adjoint* static eigenvalue problem for the initial critical state at *t*=0:

$$\mathbf{M}^{*}(\vec{r},0)\boldsymbol{\Psi}^{*}\left(\vec{r},0\right)=\mathbf{F}^{*}(\vec{r},0)\boldsymbol{\Psi}^{*}\left(\vec{r},0\right) \quad . \tag{137}$$

The adjoint problem differs from the usual *direct* problem in that all operators are replaced by

their *adjoint* counterparts. The *adjoint* $A^*$ of an operator *A* is the operator which, for any arbitrary vectors $\mathbf{\Phi}(\vec{r},t)$ and $\mathbf{\Psi}(\vec{r},t)$, satisfies:

$$\left\langle \mathbf{\Phi}, A\mathbf{\Psi} \right\rangle = \left\langle A^*\mathbf{\Phi}, \mathbf{\Psi} \right\rangle . \tag{138}$$

The reactivity at time *t* can therefore be expressed as:

$$\rho(t) = \frac{\left\langle \mathbf{\Psi}^*(\vec{r},0), \mathbf{F}(\vec{r},t)\mathbf{\Psi}(\vec{r},t) \right\rangle - \left\langle \mathbf{\Psi}^*(\vec{r},0), \mathbf{M}(\vec{r},t)\mathbf{\Psi}(\vec{r},t) \right\rangle}{\left\langle \mathbf{\Psi}^*(\vec{r},0), \mathbf{F}(\vec{r},t)\mathbf{\Psi}(\vec{r},t) \right\rangle} . \tag{139}$$

The remaining point-kinetics parameters can be expressed similarly using the initial adjoint as the weight function.

An additional result from perturbation theory states that when the adjoint function is used as the weight function, the reactivity resulting from small perturbations applied to an initially critical reactor can be calculated as:

$$\rho(t) = \frac{\left\langle \mathbf{\Psi}^*(\vec{r},0), \delta\mathbf{F}(\vec{r},t)\mathbf{\Psi}(\vec{r},0) \right\rangle - \left\langle \mathbf{\Psi}^*(\vec{r},0), \delta\mathbf{M}(\vec{r},t)\mathbf{\Psi}(\vec{r},0) \right\rangle}{\left\langle \mathbf{\Psi}^*(\vec{r},0), \mathbf{F}(\vec{r},0)\mathbf{\Psi}(\vec{r},0) \right\rangle} , \tag{140}$$

where the $\delta$ symbols represent *perturbations* (changes) in the respective operators with respect to the initial critical state. Equation (140) offers a simpler way of calculating the reactivity than Eq. (139) because it does not require recalculation of the shape vector at each time *t*. Note that, within first-order of approximation, the calculated reactivity is also equal to the static reactivity at time *t*, defined as:

$$\rho(t) = 1 - \frac{1}{k_{eff}(t)} . \tag{141}$$

In fact, perturbation theory can also be used to calculate the (static) reactivity when the initial unperturbed state is not critical. In that case, the change in reactivity is calculated as:

$$\Delta\rho = \frac{1}{k_{eff}^0} - \frac{1}{k_{eff}} = \frac{\left\langle \mathbf{\Psi}_0^*, \frac{1}{k_{eff}^0}\delta\mathbf{F}\mathbf{\Psi}_0 \right\rangle - \left\langle \mathbf{\Psi}_0^*, \delta\mathbf{M}\mathbf{\Psi}_0 \right\rangle}{\left\langle \mathbf{\Psi}_0^*, \mathbf{F}\mathbf{\Psi}_0 \right\rangle} . \tag{142}$$

In Eq. (142), the "0" subscript or superscript denotes the unperturbed state. Finally, for one-energy-group representations, the direct flux and the adjoint function are equal. It follows that in a one-group representation, the reactivity at time *t* can be expressed as:

$$\rho(t) = \frac{\left\langle \mathbf{\Psi}(\vec{r},0), \delta\mathbf{F}(\vec{r},t)\mathbf{\Psi}(\vec{r},0) \right\rangle - \left\langle \mathbf{\Psi}(\vec{r},0), \delta\mathbf{M}(\vec{r},t)\mathbf{\Psi}(\vec{r},0) \right\rangle}{\left\langle \mathbf{\Psi}(\vec{r},0), \mathbf{F}(\vec{r},0)\mathbf{\Psi}(\vec{r},0) \right\rangle} =$$

$$\frac{\int_{V_{core}} \Phi^2(\vec{r},0)\left(\delta\nu\Sigma_f - \delta\Sigma_a\right)dV}{\int_{V_{core}} \Phi^2(\vec{r},0)\nu\Sigma_f dV} . \tag{143}$$

More generally, the static reactivity change between any two states, which is the equivalent of Eq. (142), can be expressed using one-group diffusion theory as:

$$\Delta\rho = \frac{1}{k_{eff}^0} - \frac{1}{k_{eff}} = \frac{\int\limits_{V_{core}} \Phi_0^2(\vec{r})\left(\frac{1}{k_{eff}^0}\delta\nu\Sigma_f - \delta\Sigma_a\right)dV}{\int\limits_{V_{core}} \Phi_0^2(\vec{r})\nu\Sigma_{f0}dV} \quad . \tag{144}$$

## 6.2 Device Reactivity Worth

Reactivity devices are devices, usually rods, made of material with high neutron-absorption cross section. By inserting or removing a device, the reactivity of the reactor can be changed, and hence the power can be decreased or increased. The *reactivity worth* of a device is defined as the difference between the reactivity of the core with the device inserted and the reactivity of the same core with the device removed. Looking at this situation through a perturbation-theory lens, the reactor without the reactivity device can be regarded as the unperturbed system, and the reactor with the reactivity device can be regarded as the perturbed system. Perturbation theory offers interesting insights into reactivity worth. Considering a device that is inserted into a critical reactor and which, after insertion, occupies volume $V_d$ in the reactor, according to the perturbation formula for reactivity, the reactivity worth of the device is:

$$\Delta\rho_d = \frac{1}{k_{eff}^0} - \frac{1}{k_{eff}^d} = \frac{\int\limits_{V_d} \Phi_0^2\left[\frac{1}{k_{eff}^0}\left(\nu\Sigma_{fd} - \nu\Sigma_{f0}\right) - \left(\Sigma_{ad} - \Sigma_{a0}\right)\right]dV}{\int\limits_{V_{core}} \Phi_0^2\nu\Sigma_{f0}dV} \quad . \tag{145}$$

Note that the integral in the numerator is over the device volume only and that the integral in the denominator does not change as the device moves, thus simplifying the calculations. Moreover, if two devices are introduced, their combined reactivity worth is:

$$\Delta\rho_{d1+d2} = \frac{\int\limits_{V_{d1}} \Phi_0^2\left[\frac{1}{k_{eff}^0}\left(\nu\Sigma_{fd1} - \nu\Sigma_{f0}\right) - \left(\Sigma_{ad1} - \Sigma_{a0}\right)\right]dV}{\int\limits_{V_{core}} \Phi_0^2\nu\Sigma_{f0}dV} + \frac{\int\limits_{V_{d2}} \Phi_0^2\left[\frac{1}{k_{eff}^0}\left(\nu\Sigma_{fd2} - \nu\Sigma_{f0}\right) - \left(\Sigma_{ad2} - \Sigma_{a0}\right)\right]dV}{\int\limits_{V_{core}} \Phi_0^2\nu\Sigma_{f0}dV} =$$

$$\Delta\rho_{d1} + \Delta\rho_{d2} \quad . \tag{146}$$

The interpretation of this equation is that as long as devices are not too close together and do not have too large reactivity worths (so that the assumptions of perturbation theory remain valid), their reactivity worths are additive.

# 7 Fission-Product Poisoning

Poisons are nuclides with large absorption cross sections for thermal neutrons. Some poisons are introduced intentionally to control the reactor, such as B or Gd. Some poisons are produced as fission products during normal reactor operation. Xe and Sm are the most important of these.

## 7.1 Effects of Poisons on Reactivity

The effect of poisons on a reactor will be studied for a simple model of a homogeneous reactor modelled using one-group diffusion theory. For such a reactor, in a one-energy-group formalism:

$$k_{eff}^0 = \frac{\nu\Sigma_f}{\Sigma_{a0} + DB_g^2} . \tag{147}$$

Uniform concentration

If a poison such as Xe with microscopic cross section $\sigma_{ax}$ is added with a uniform concentration (number density) *X*, the macroscopic absorption cross section increases by:

$$\Sigma_{aX} = X\sigma_{aX} . \tag{148}$$

The total macroscopic absorption cross section is now:

$$\Sigma_a = \Sigma_{a0} + \Sigma_{aX} , \tag{149}$$

and the new effective multiplication constant is:

$$k_{eff} = \frac{\nu\Sigma_f}{\Sigma_a + DB_g^2} = \frac{\nu\Sigma_f}{\Sigma_{a0} + \Sigma_{aX} + DB_g^2} . \tag{150}$$

Addition of the poison induces a change in reactivity:

$$\Delta\rho = \rho - \rho_0 = \left(1 - \frac{1}{k_{eff}}\right) - \left(1 - \frac{1}{k_{eff}^0}\right) = \frac{1}{k_{eff}^0} - \frac{1}{k_{eff}} =$$
$$\frac{\Sigma_{a0} + DB^2}{\nu\Sigma_f} - \frac{\Sigma_{a0} + \Sigma_{aX} + DB^2}{\nu\Sigma_f} =$$
$$-\frac{\Sigma_{aX}}{\nu\Sigma_f} = -\frac{X\sigma_{aX}}{\nu\Sigma_f} . \tag{151}$$

To calculate the reactivity inserted by the poison, the concentration of poison nuclei, *X*, must first be determined.

Non-uniform concentration

In the case of non-uniform poison concentration, the perturbation formula for reactivity can be

used:

$$\Delta\rho = \frac{1}{k_0} - \frac{1}{k} = -\frac{\int_V \Phi^2(\vec{r})\delta\Sigma_a(\vec{r})dV}{\int_V \Phi^2(\vec{r})\nu\Sigma_f(\vec{r})dV} = \frac{\int_V \Phi^2(\vec{r})\Sigma_{aX}(\vec{r})dV}{\int_V \Phi^2(\vec{r})\nu\Sigma_f(\vec{r})dV} = -\frac{\int_V \Phi^2(\vec{r})X(\vec{r})\sigma_{aX}dV}{\int_V \Phi^2(\vec{r})\nu\Sigma_f(\vec{r})dV} \quad . \qquad (152)$$

It can easily be seen that if the distribution is uniform, the previous formula is recovered:

$$\Delta\rho = \frac{1}{k_0} - \frac{1}{k} = -\frac{\Sigma_{aX}\int_V \Phi^2(\vec{r})dV}{\nu\Sigma_f \int_V \Phi^2(\vec{r})dV} = -\frac{\Sigma_{aX}}{\nu\Sigma_f} \quad . \qquad (153)$$

In the next sub-section, specific aspects of fission-product poisoning will be illustrated for the case of Xe.

## 7.2 Xenon Effects

### 7.2.1 $^{135}$Xe production and destruction

The mechanisms of $^{135}$Xe production and destruction are illustrated in Fig. 2.

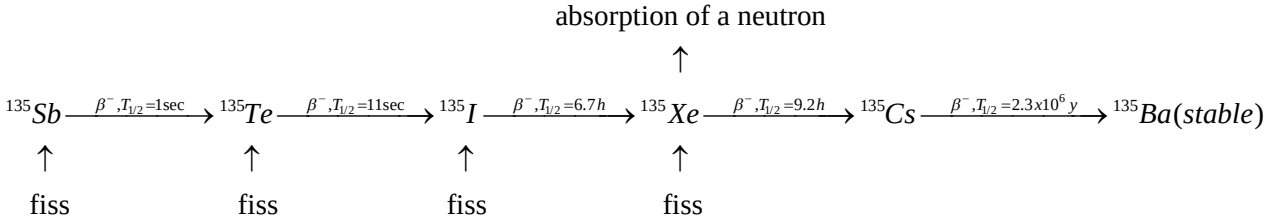


**Figure 2 $^{135}$Xe production and destruction mechanisms**

Because $^{135}$Sb decays very rapidly into $^{135}$Te, which in turn decays very rapidly into $^{135}$I, as an approximation, $^{135}$I can be considered to be produced directly from fission. Because $^{135}$Cs has a very long half-life, as an approximation, it can be considered stable. As a consequence of these approximations, a simplified $^{135}$Xe production and destruction scheme can be used, as illustrated in Fig. 3.

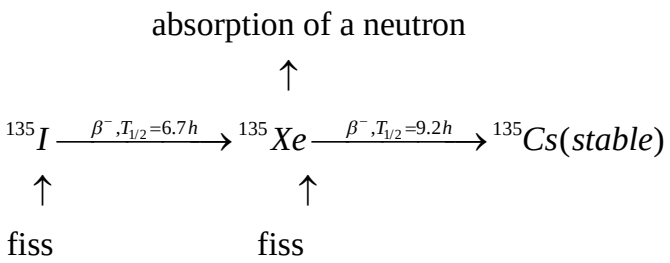


**Figure 3 Simplified $^{135}$Xe production and destruction mechanisms**

### 7.2.2 Determining the Xe concentration

To find the numerical density of Xe nuclei, the balance equation for iodine nuclei is first written

as:

$$\frac{dI}{dt} = \gamma_I \Sigma_f \Phi \ - \lambda_I I \quad , \tag{154}$$

where $\gamma$ is called the fission product yield and equals the average number of I nuclides created per fission. The balance equation for Xe nuclei can be subsequently written as:

$$\frac{dX}{dt} = \lambda_I I + \gamma_X \Sigma_f \Phi \ - \lambda_X X - \sigma_{aX} \Phi \ X \quad . \tag{155}$$

Steady-state conditions

Equilibrium conditions are attained after the reactor operates for a very long time ($\infty$) at a steady-state flux level $\Phi_{ss}$. Under equilibrium conditions, the concentration of I nuclei is easily found to be:

$$I_\infty = \frac{\gamma_I \Sigma_f \Phi_{ss}}{\lambda_I} \quad . \tag{156}$$

Similarly, the Xe concentration can be determined as:

$$X_\infty = \frac{\lambda_I I_\infty + \gamma_X \Sigma_f \Phi_{ss}}{\lambda_X + \sigma_{aX} \Phi_{ss}} = \frac{(\gamma_I + \gamma_X) \Sigma_f \Phi_{ss}}{\lambda_X + \sigma_{aX} \Phi_{ss}} \quad . \tag{157}$$

Note that both I and Xe concentrations depend on flux level. However, whereas the I concentration increases indefinitely with flux level, the Xe concentration levels off, and it can, at most, become equal to:

$$X_{\infty - \max} = \frac{(\gamma_I + \gamma_X) \Sigma_f}{\sigma_{aX}} \quad . \tag{158}$$

The Xe macroscopic absorption cross section is:

$$\Sigma_{aX} = X_\infty \sigma_{aX} = \frac{(\gamma_I + \gamma_X) \Sigma_f \Phi_{ss} \sigma_{aX}}{\lambda_X + \sigma_{aX} \Phi_{ss}} \quad . \tag{159}$$

Using the notation:

$$\Phi_X = \frac{\lambda_X}{\sigma_{aX}} = 0.770 \times 10^{13} cm^{-2} \sec^{-1} \quad , \tag{160}$$

the Xe macroscopic absorption cross section can be rewritten as:

$$\Sigma_{aX} = \frac{(\gamma_I + \gamma_X) \Sigma_f \Phi_{ss}}{\Phi_X + \Phi_{ss}} \quad . \tag{161}$$

If Xe is assumed to be uniformly distributed, then its resulting reactivity worth is:

$$\rho_{Xe} = -\frac{\Sigma_{aX}}{\nu\Sigma_f} = \frac{1}{\nu\Sigma_f}\frac{(\gamma_I+\gamma_X)\Sigma_f\Phi_{ss}}{\Phi_X+\Phi_{ss}} = -\frac{\left(\gamma_I+\gamma_X\right)}{\nu}\frac{\Phi_{ss}}{\Phi_X+\Phi_{ss}} \quad . \tag{162}$$

For high reactor fluxes, in the case where $\Phi_{ss} >> \Phi_X$, $\Phi_X$ can be neglected in the denominator, and the reactivity becomes independent of the flux level and equal to its maximum value of:

$$\rho_{Xe} \cong -\frac{\gamma_I+\gamma_X}{\nu} \quad . \tag{163}$$

This is to be expected given that the Xe concentration has been found to "saturate" with increased flux. The reactivity expressed in Eq. (163) is nothing but the corresponding reactivity for the maximum Xe concentration shown in Eq. (158).

If, on the contrary, the flux is very low, in the case where $\Phi_{ss} << \Phi_X$, then $\Phi_{SS}$ can be neglected in the denominator, and the Xe equivalent reactivity increases linearly with flux level:

$$\rho_{Xe} = -\frac{\left(\gamma_I+\gamma_X\right)}{\nu}\frac{\Phi_{ss}}{\Phi_X} \quad . \tag{164}$$

Xe load after shutdown: reactor dead time

If, the reactor is shut down ($\Phi = 0$), I and Xe production from fission ceases, as well as Xe destruction through neutron absorption. The concentration of I begins to decrease exponentially due to decay. If the I concentration at the time of shutdown is $I_0$, the I concentration as a function of time can be expressed simply as:

$$I(t) = I_0 e^{-\lambda_I t} \quad . \tag{165}$$

Substituting this expression into the Xe balance equation and setting the flux to zero leads to the following expression:

$$\frac{dX}{dt} = \lambda_I I_0 e^{-\lambda_I t} - \lambda_X X \quad . \tag{166}$$

Denoting the Xe concentration at the time of shutdown by $X_0$, the solution can be written as:

$$X(t) = X_0 e^{-\lambda_X t} + \frac{\lambda_I I_0}{\lambda_I - \lambda_X}(e^{-\lambda_X t} - e^{-\lambda_I t}) \quad . \tag{167}$$

If the reactor is shut down after operating for a long time at steady state, the resulting Xe concentration is:

$$X(t) = \frac{(\gamma_1+\gamma_X)\Sigma_f\Phi_{ss}}{\lambda_X+\sigma_{aX}\Phi_{ss}} e^{-\lambda_X t} + \frac{\gamma_I\Sigma_f\Phi_{ss}}{\lambda_1-\lambda_X}(e^{-\lambda_X t} - e^{-\lambda_I t}) \quad . \tag{168}$$

The equivalent reactivity for uniformly distributed Xe (and assuming that the reactor was shut down after operating for a long time at steady state) is:

$$\rho = -\frac{1}{\nu}\left[\frac{(\gamma_I+\gamma_X)\Phi_{ss}}{\Phi_X+\Phi_{ss}}e^{-\lambda_X t}+\frac{\gamma_I\Phi_{ss}}{\Phi_I-\Phi_X}(e^{-\lambda_x t}-e^{-\lambda_I t})\right], \tag{169}$$

where:

$$\Phi_I=\frac{\lambda_I}{\sigma_{aX}}=1.055\times10^{13}\,cm^{-2}\,\sec^{-1}. \tag{170}$$

The Xe concentration, and consequently the Xe reactivity worth after shutdown, increases at first because Xe continues to be produced by decay of I, whereas consumption is now reduced in the absence of Xe destruction by neutron absorption. After a while, however, the Xe concentration reaches a maximum, starts decreasing, and eventually approaches zero. This behaviour is shown in Fig. 4, which shows the Xe reactivity worth after shutdown from full power.

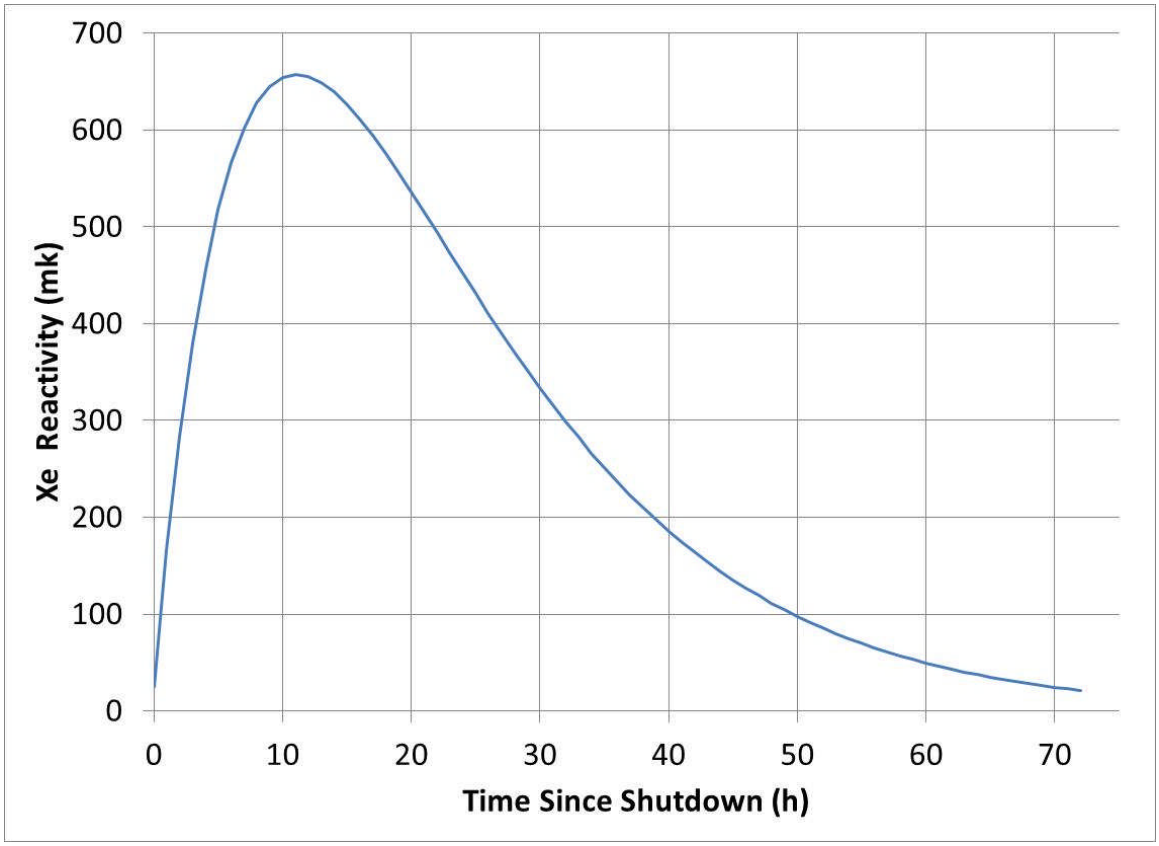


**Figure 4 $^{135}$Xe reactivity worth after shutdown**

Because the Xe reactivity (or *load*) increases after shutdown, a reactor that was critical at the time of shutdown subsequently becomes sub-critical and cannot be restarted until the Xe load drops back to a value close to its steady-state level. The time during which the reactor cannot be restarted due to increased Xe load after shutdown is known as reactor "dead time". Given the Xe half-life of approximately nine hours, the reactor dead time, which spans several half-lives, is of the order of 1.5–2 days. Some reactors have systems to compensate for some of the shutdown Xe load, in the form of reactivity devices that are inserted in the core during normal

steady-state reactor operation. As Xe builds up after shutdown, removal of these devices can counterbalance the Xe reactivity load, enabling the reactor to be brought to critical and re-started. Adjuster rods in the CANDU reactor can serve this purpose, but only up to 30 minutes after shutdown. Beyond 30 minutes, the Xe load becomes larger than the adjuster-rod reactivity worth. Because the Xe load increases with the neutron flux, Xe-poison dead time generally affects only high-power reactors.

# 8 Reactivity Coefficients and Feedback

Macroscopic cross sections can change as a consequence of various parameters, and in turn, these changes induce changes in $k_{eff}$ and hence in reactivity. The usual parameters that influence reactivity are:

- fuel temperature
- coolant temperature
- moderator temperature
- coolant density.

Changes in reactivity induced by changes in any such parameter are referred to as the *reactivity effect* of the respective parameter. For example, the reactivity change induced by a change in fuel temperature is called the *fuel-temperature reactivity effect*. The derivative of the reactivity with respect to any of the parameters, with the others kept constant (i.e., the partial derivative), is called the *reactivity coefficient* of that parameter. To illustrate this, assume that all parameters are kept constant with the exception of one, e.g., fuel temperature, which is varied. Assume further that reactivity is plotted as a function of the variable parameter, in this case fuel temperature. The plot in question would be a plot of $\rho(T_f)$. If a certain fuel temperature $T_{f0}$ is taken as a reference, then the effect on reactivity of deviations from $T_{f0}$ can be calculated, namely $\Delta\rho(T_f) = \rho(T_f) - \rho(T_{f0})$. $\Delta\rho(T_f)$ is called the *fuel-temperature reactivity effect*. The derivative of the reactivity with respect to the fuel temperature, namely $\alpha_{T_f}(T_f) = \dfrac{d\rho(T_f)}{dT_f}$, is called the *fuel-temperature coefficient*. Of course, because reactivity also depends on other parameters, it becomes clear that this derivative should be a partial derivative. In general, if the reactivity depends on several parameters,

$$\rho = \rho(p_1, \ldots p_n), \tag{171}$$

then the reactivity coefficient with respect to a parameter $p_i$ is defined as the partial derivative of the reactivity with respect to that parameter:

$$\alpha_{p_i} = \frac{\partial\rho(p_1, \ldots p_n)}{\partial p_i}. \tag{172}$$

Of course, in reality, several parameters may vary simultaneously, and their variations may be impossible to decouple. For example, the moderator density also varies when the moderator temperature varies, due to thermal expansion. In this case, the single-parameter reactivity coefficients are somewhat artificial, and additional combined reactivity coefficients can be defined, which are of more practical use. For example, in the case of simultaneous variation of

both moderator temperature $T_m$ and moderator density $d_m$, a much more useful quantity would be:

$$\alpha_{T_m(d_m)} = \frac{d\rho}{dT_m} = \frac{\partial\rho(T_m, d_m)}{\partial T_m} + \frac{\partial\rho(T_m, d_m)}{\partial d_m}\frac{dd_m}{dT_m}, \tag{173}$$

where the derivative $\frac{dd_m}{dT_m}$ is specified by the thermal expansion law.

Such coefficients are called *combined* reactivity coefficients. One very useful combined coefficient of this kind is the *power coefficient of reactivity* (PCR), which is defined as:

$$PCR = \frac{d\rho}{dP} = \sum_{i=1}^{n} \frac{\partial\rho(p_1 \ldots p_n)}{\partial p_i}\frac{dp_i}{dP}, \tag{174}$$

where the $p_i$ are all the parameters which change with power and on which the reactivity ultimately depends, such as fuel temperature, coolant temperature, coolant density, and so on.

Because the core parameters depend on the flux level and because they influence the reactivity, which in turn influences the flux, the reactivity coefficients are said to express the feedback which describes the connection between the flux level and the cross-section values. When such feedback is accounted for during a transient by recalculating the cross sections and the reactivity as functions of the flux level (and hence of power level), it is said that kinetics calculations with feedback, or *dynamic* calculations, are being performed.

This sub-section will close with a presentation of a few alternate expressions for reactivity coefficients. If reactivity is expressed using the effective multiplication constant, then the reactivity coefficient can be expressed as follows:

$$\alpha_P = \frac{d\rho(p)}{dp} = \frac{d}{dp}\left(1 - \frac{1}{k(p)}\right) = \frac{1}{k^2(p)}\frac{dk(p)}{dp}. \tag{175}$$

For reactors close to critical ($k \cong 1$), this can be processed to yield:

$$\alpha_P = \frac{1}{k^2(p)}\frac{dk(p)}{dp} \cong \frac{1}{k(p)}\frac{dk(p)}{dp}. \tag{176}$$

The last form can also be expressed as:

$$\frac{1}{k(p)}\frac{dk(p)}{dp} = \frac{d}{dp}\ln\left[k(p)\right]. \tag{177}$$

The last two expressions are often used to calculate reactivity coefficients.

# 9 CANDU-Specific Features

Given the presentation of the basic concepts of reactor kinetics in previous sections, this section will be devoted to presenting how some of these concepts apply to CANDU reactors.

## 9.1 Photo-Neutrons: Additional Delayed-Neutron Groups

CANDU reactors are heavy-water cooled and moderated. In such reactors, neutrons can be produced by the interaction of gamma rays (with a minimum energy of 2.22 MeV) with deuterium:

$$ {}_1^2D + \gamma \rightarrow {}_1^1H + n \,. \tag{178}$$

Because gamma rays can be emitted by fission products with certain delays, the process is very similar to that through which a "true" delayed neutron is emitted by an emitter which is the daughter of a precursor. However, note than not all gamma rays will interact by photo-neutron emission. Effective precursor concentrations can be defined for the photo-neutrons, such that the photo-neutron production rate density is equal to:

$$ S_{pn} = \lambda_{pn} C_{pn} \,. \tag{179}$$

The term *effective* photo-neutron precursor concentration is used because the *effective* concentrations must also account for the fact that not all emitted gamma rays will result in the production of a photo-neutron and that the fraction of photo-neutrons emitted depends on the geometrical arrangement of the core lattice. Photo-neutron precursors can be grouped by their decay constant, similarly to "real" precursors. It is customary to use 11 photo-neutron groups, for a total of 17 delayed-neutron groups. Once the photo-neutron groups have been defined, photo-neutrons are treated no differently than regular delayed neutrons in the kinetics calculations.

## 9.2 Values of Kinetics Parameters in CANDU Reactors: Comparison with LWR and Fast Reactors

Deuterium has a much smaller neutron-absorption cross section than hydrogen. Consequently, CANDU reactors have a better thermalized spectrum and hence a much longer generation time (and lifetime) than light-water reactors and even longer compared to fast reactors. The typical generation time of a CANDU core is approximately 1 ms, compared to 0.05 ms for an LWR core. This makes CANDU transients "slower" than LWR transients. Reactivities close in value to the delayed-neutron fraction induce less peak power transients in a CANDU core than in an LWR core. The difference is even larger when comparison is made with fast reactors.

## 9.3 CANDU Reactivity Effects

CANDU reactivity effects depend on the specific characteristics of the CANDU lattice. Of particular interest is the *coolant density effect*, also known as the coolant void reactivity effect, which is absent in other types of reactors which do not separate the coolant from the moderator. Plots of the various effects are shown in Figs. 5 to 8 for fresh fuel.

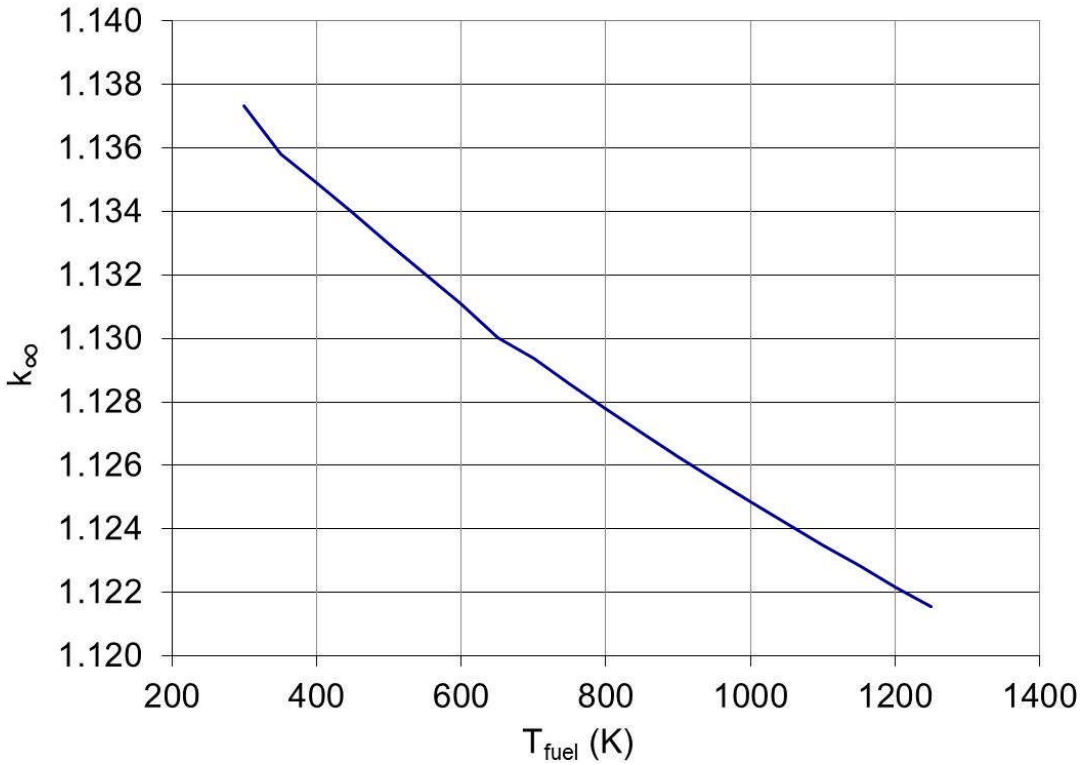


**Figure 5 CANDU fuel-temperature effect**

The decrease in reactivity with increased fuel temperature is due primarily to an increase in resonance absorption due to Doppler resonance broadening.

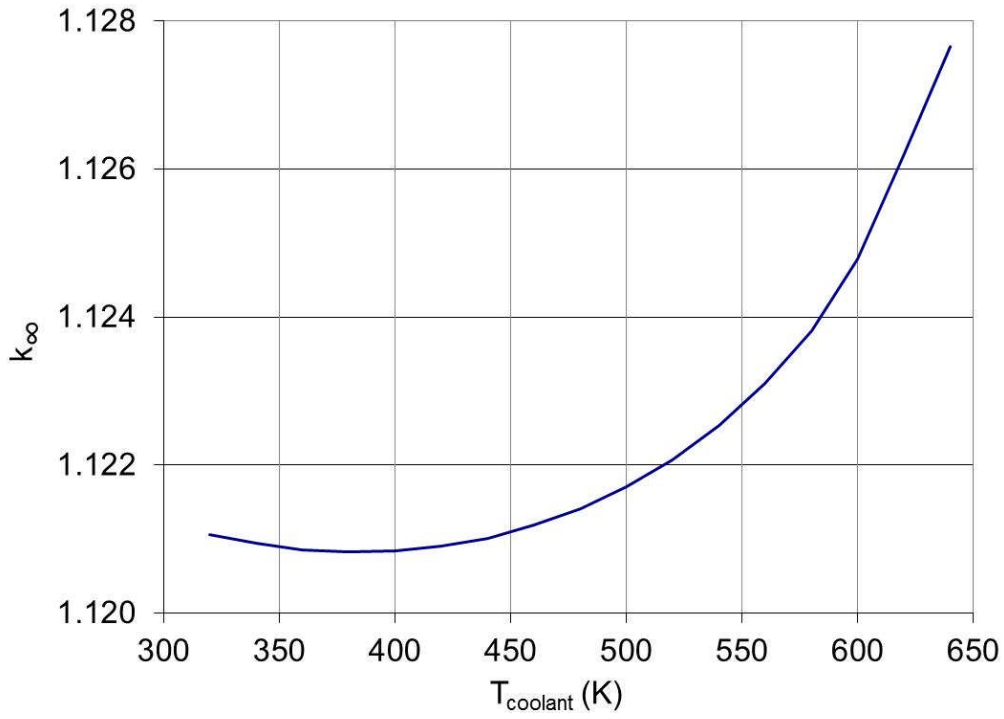


**Figure 6 CANDU coolant-temperature effect**

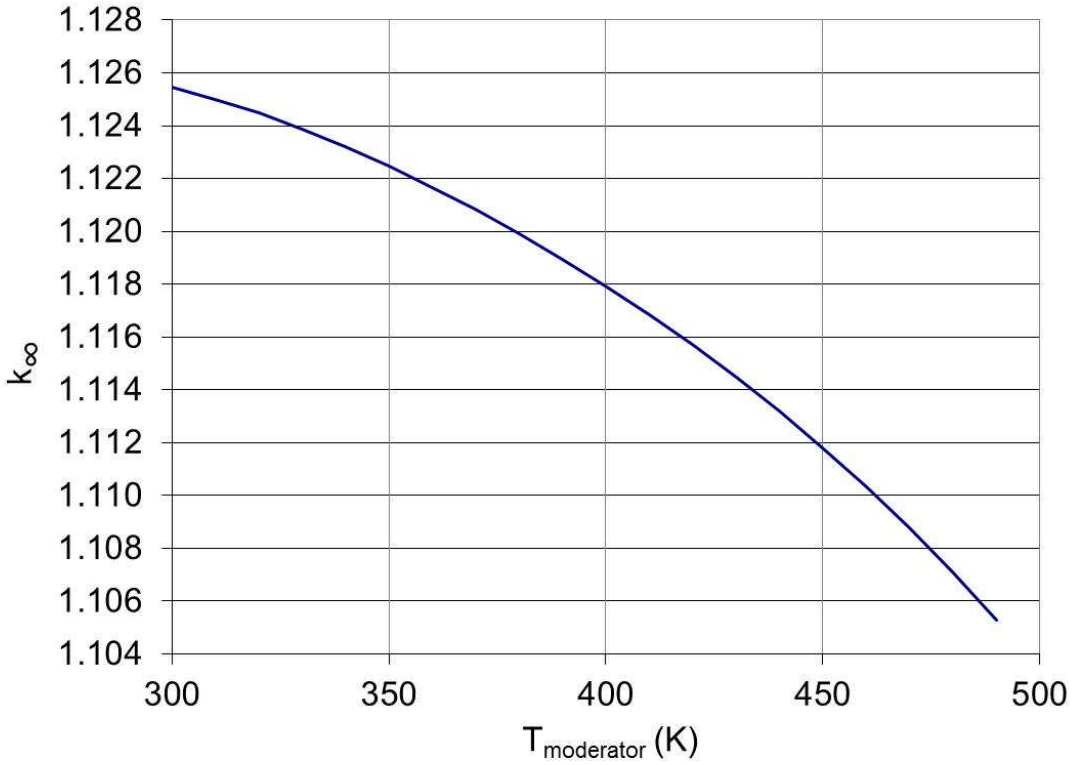


**Figure 7 CANDU moderator-temperature effect**

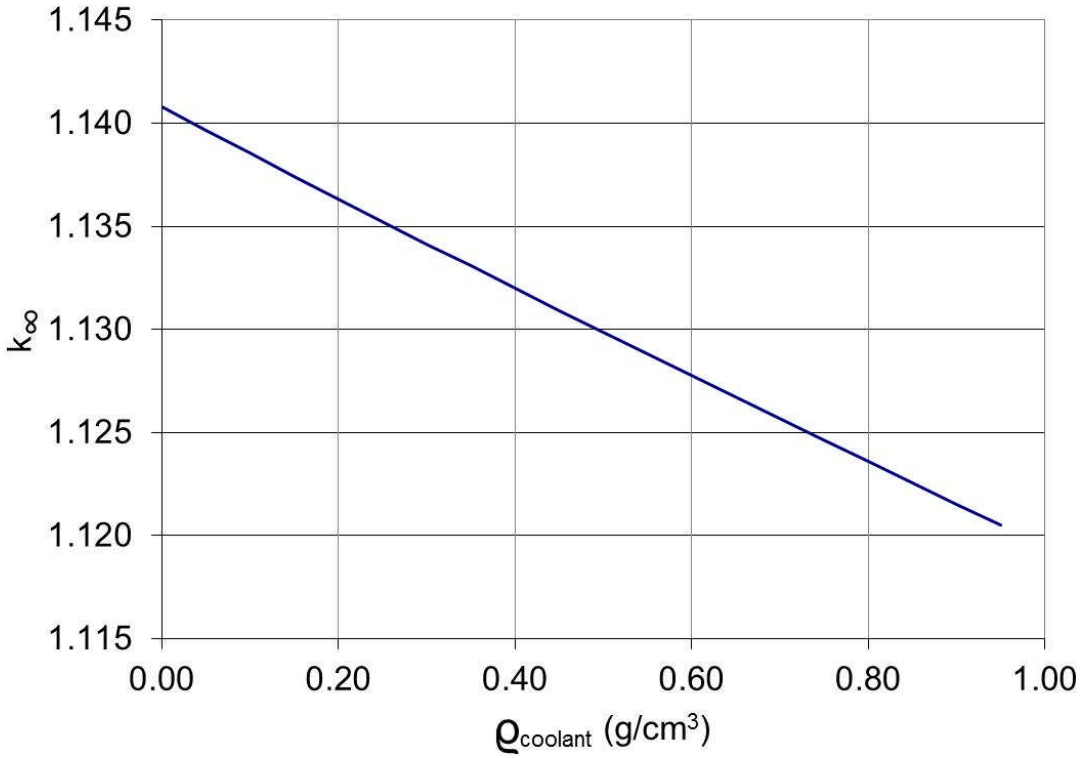


**Figure 8 CANDU coolant-density effect**

It is apparent that reactivity decreases with coolant density, which means that it increases with void fraction. This effect occurs primarily because when coolant is lost, effective moderation is still possible due to the moderator in the calandria vessel. Overall, the reduced slowing-down in

the coolant leads to reduced resonance absorption and an increased fast fission rate. Both these phenomena increase reactivity when coolant is lost. As fuel burns, a mitigating factor appears in the form of the low-lying Pu fission resonance, which begins to play a role as Pu is created. Reduced upscattering in the coolant reduces fissions in the low-lying Pu resonance and hence reduces reactivity, although not enough to make it negative.

## 9.4 CANDU Reactivity Devices

As in any reactor, power in CANDU reactors is controlled by controlling reactivity. In turn, reactivity is controlled by means of reactivity devices which can be inserted into or removed from the core. By inserting or removing reactivity devices from the core, the absorption rate is varied; hence, reactivity can be varied, and power can be increased or decreased, or the reactor can be completely shut down. Reactivity devices in CANDU reactors come under the control either of the Reactor Regulating System (RRS) or of one of the two independent shutdown systems (SDS1 and SDS2).

For a CANDU 6, the reactivity devices under the control of the RRS are as follows:

- 14 liquid-zone-control compartments ($H_2O$-filled)
- 21 adjuster rods
- 4 mechanical control absorbers
- moderator “poison”.

The reactivity devices under the control of the shutdown systems are:

SDS-1: 28 cadmium shutoff rods which fall into the core from above

SDS-2: high-pressure Gd-poison injection into the moderator through six horizontally oriented nozzles.

The reactivity worth of these reactivity devices is shown in Table 2.

**Table 2 CANDU reactivity device worth**

| Function | Device | Total Reactivity Worth (mk) | Maximum Reactivity Rate (mk/s) |
|---|---|---|---|
| Control | 14 liquid zone con-trollers | 7 | ± 0.14 |
| Control | 21 adjusters | 15 | ± 0.10 |
| Control | 4 mechanical control absorbers | 10 | ± 0.075 (driving)<br>-3.5 (dropping) |
| Control | moderator poison | – | -0.01 (extracting) |
| Safety | 28 shut-off units | 80 | -50 |
| Safety | 6 poison-injection nozzles | > 300 | -50 |

# 10 Summary of Relationship to Other Chapters

Chapter 5 relies on knowledge acquired from Chapter 4, Reactor Statics and indirectly on knowledge from Chapter 3, Nuclear Processes and Neutron Physics.

# 11 Problems

1. $10^{20}$ nuclei of $^{235}U$ undergo fission with a delayed neutron yield of 0.0125 and a delayed neutron fraction of 0.005.
   a) What is the total neutron yield?
   b) How many precursors are produced?
   c) How many emitters are eventually produced?

2. A radioactive waste site consists of two cylindrical tanks that contain liquid waste in the form of fissile material in an aqueous solution. The tanks are in the form of cubes with the side equal to 1m. The neutronic properties of the radioactive waste are: SA=0.00100cm$^{-1}$, D=1cm, nu=2.5, SF=0.00158 cm$^{-1}$, v=2200 m/s. Assume all fission neutrons are prompt.
   a) Calculate the reactivity, generation time and neutron life time for one of the tanks.
   b) The site manager decides to save space and money, by storing the content of both tanks in a larger cubical tank, with side $\sqrt[3]{2}m$. Calculate the reactivity, generation time and neutron life time for the new tank. Comment on the result.

3. A thin foil made of a mixture of isotopes one of which is fissile has a fission macroscopic cross section equal to 0.001 cm$^{-1}$. When fissioning, the fissile isotope produces two types of precursors: one with yield 0.05 and a half-life of one minute, and another with yield 0.03 and a half-life of two minutes. The sample is subjected to a pulse of neutrons at t=0, and to another pulse of neutrons at t=90s. The first pulse has a fluence of $10^8$ n/cm$^2$, and the second pulse has a fluence of $5x10^7$ n/cm$^2$. What is the total number of precursors at t=5 minutes?
   **Note:** Assume that the number of nuclei that react with neutrons (through fission or otherwise) is negligible compared to the original number of nuclei present in the sample, and that neutrons emitted from the sample do not interact in the sample.

4. A thin-foil of fissile material is irradiated uniformly in a neutron flux of $10^8$ n/cm$^2$/s, starting at t=0. There are $10^{22}$ fissile nuclei in the sample and the fission microscopic cross section is 2000b. There is only one group of delayed neutrons. The total neutron yield per fission is 2.33. The delayed neutron fraction is 0.001 and the half-life of precursors is one minute. The number of neutrons emitted by the sample per second is measured with a neutron detector.
   a) What does the detector indicate 20 minutes after the start of the irradiation?
   b) At 25 minutes the irradiation stops. What does the detector indicate two minutes later?
   **Note:** Assume that the number of nuclei that react with neutrons (through fission or otherwise) is negligible compared to the original number of nuclei present in the sample, and that neutrons emitted from the sample do not interact in the sample.

5. Consider a homogeneous nuclear reactor for which all neutrons are born prompt. The reactor is cubical, with side equal to 4m. The neutronic parameters of the reactor are:
   $D = 1cm$
   $\Sigma_a = 0.003cm^{-1}$
   $\nu = 2.5$
   $\overline{v} = 2200m/s$
   The reactor is initially critical, operating at 3000 MW fission power. The energy liberated per fission is approximately 200MeV. The extrapolated size of the reactor can be approximated by its physical size.
   a) Find $\Sigma_f$.
   b) Calculate the volume-integrated flux in the reactor.
   c) Calculate total neutron population in the reactor.
   d) Calculate the neutron generation time and life time.

6. Consider a slab homogeneous reactor (infinite in the y and z directions and finite in the x direction) extending from -2m to 2m in the x direction, and with the following parameters:
   $D = 1cm$
   $\Sigma_a = 0.003cm^{-1}$
   $\nu = 2.5$
   Assume that the physical length and the extrapolated length of the reactor can be approximated to be equal.
   The reactor is initially critical.
   a) Find the fission cross section.
   b) A control plate, 1 cm thick (and extending to infinity in the y and z directions, just like the reactor) is introduced at x=1m. The neutronic parameters of the control plate are:
      $D = 1cm$
      $\Sigma_a = 0.01cm^{-1}$
      $\Sigma_f = 0cm^{-1}$
   c) Assuming the control plate is thin-enough so that the unperturbed flux across it can be assumed to be constant and equal to the value at the center of the plate, calculate the reactivity of the reactor after the introduction of the control plate.
   d) **Note:** When a plate is inserted in the reactor, its material **displaces** (that is replaces) the unperturbed reactor material at the position of the plate.

7. Consider a reactor with 6 delayed neutron groups, with the following parameters.
   $\bar{v} = 2200\ m/s$
   $\Lambda = 0.001\ s$
   $\rho = 0.000$
   $\nu = 2.5$
   $\beta_1 = 0.0010$
   $\beta_2 = 0.0012$
   $\beta_3 = 0.0008$
   $\beta_4 = 0.0002$
   $\beta_5 = 0.0011$
   $\beta_6 = 0.0014$
   $\lambda_1 = 0.001\ s^{-1}$
   $\lambda_2 = 0.005\ s^{-1}$
   $\lambda_3 = 0.010\ s^{-1}$
   $\lambda_4 = 0.015\ s^{-1}$
   $\lambda_5 = 0.050\ s^{-1}$
   $\lambda_6 = 0.100\ s^{-1}$
   The reactor operates in steady state at 2000 MW. The energy per fission is approximately 200MeV.
   What is the total delayed neutron fraction?
   What is the total delayed neutron yield?
   What is the precursor population for each of the groups 1 to 6?
   What is the total delayed neutron source?

8. Consider a homogeneous nuclear reactor with one delayed neutron group. The reactor is cubical, with side equal to 4m. The neutronic parameters of the reactor are:
   $D = 1cm$
   $\Sigma_a = 0.003cm^{-1}$
   $\Sigma_f = 0.00136cm^{-1}$
   $\nu = 2.5$
   $\bar{v} = 2200m/s$
   $\beta = 0.005$
   $\lambda = 0.2s^{-1}$
   Assume the extrapolated size of the reactor equals its physical size.
   a) Calculate $k_{eff}$.
   b) The reactor is maintained subcritical by the addition of $^{10}B$, which has a microscopic cross section of approximately 4000b. ($1\,barn = 10^{-24}cm^2$). If $k = 0.980$, what is the number density of Boron atoms?
   c) An external neutron source is introduced into the reactor at point b). The neutron balance (point kinetics) equations are thus written:

$$\dot{n} = \frac{\rho - \beta}{\Lambda} n + \lambda C + S$$

$$\dot{C} = \frac{\beta}{\Lambda} n - \lambda C$$

where n is the total neutron population, and S is the strength of the external source (neutrons/s).

If $S = 10^6 \frac{neutrons}{s}$, what is the equilibrium (steady-state) neutron population? What is the equilibrium precursor population?

d) At t=0, the external neutron source is removed from the reactor. What are the neutron population and the precursor population 5 seconds after the removal of the external source?

9. Consider a reactor with six delayed neutron groups, with the following parameters:
   $\Lambda = 0.001s$
   $\beta = 0.005$
   The reactor is initially operating at a steady-state power of 1000MW. A control rod that was initially in the core is accidentally ejected at t=0, yielding a 2mk reactivity increase. What is the reactor power immediately after the rod ejection? Use the prompt jump approximation.

10. Consider a reactor with one delayed neutron group, with the following parameters:
    $\Lambda = 0.001s$
    $\beta = 0.005$
    $\lambda = 0.02s^{-1}$
    The reactor is initially in steady state operation at a power of 1000MW. A control rod with a reactivity worth of 2 mk is inserted in the reactor at t=0. At t=2s, a second, identical, control rod is inserted. What is the reactor power at t=4s?
    **Notes:**
    Use the prompt jump approximation.
    The prompt jump approximation is also valid when the reactor is not initially critical.

# 12 References and Further Reading

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## 13 Acknowledgements

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