# Point Kinetics

THE POINT KINETICS MODEL can be obtained directly from the space and time dependent transport equations. However these equations are too complicated to be of any practical application. The diffusion approximation, obtained by keeping only the $P_1$ terms of the spherical harmonics expansion in the angular variable of the directional flux is frequently used in neutronic analysis. This is the diffusion approximation, which we use here.

## Derivation

Point kinetics is very interesting because of the apparent simplicity of the resulting differential equations. The method is very frequently used, but the underlying difficulties in obtaining the parameters are

hidden. In spite of all this, many inherent characteristics of the dynamics of nuclear cores can be deduced from these equations. Also, these same equations provide a tool for the analysis, the comparison and the practical implementation of various numerical schemes that may eventually be used in more complex situations. An integration technique that does not pass the test of point kinetics will certainly not be used in space-time kinetics. Point kinetics can thus play the role of an experimental bench before expensive problems are attempted with more advanced methods.

The main idea behind point kinetics is to separate the flux into two factors. The first one being a shape function depending both on space and time, and a second factor depending only on time, in the following fashion

$$[\phi(\vec{r}, t)] = [S(\vec{r}, t)]T(t) \qquad \text{(EQ 73)}$$

Note that this equation for the flux does not involve any approximation, and that the equality is maintained.However, the shape function $[S(\vec{r}, t)]$ depends both on space and time in this approach.We now introduce a column vector of weight functions

$$[W(\vec{r})] = \begin{bmatrix} W_1(\vec{r}) \\ W_2(\vec{r}) \\ \cdot \\ W_G(\vec{r}) \end{bmatrix}$$

whose function will be to give rise to general equations. In effect, equation (73) presents a degree of arbitrariness in the choice of $[S(\vec{r}, t)]$ and of $T(t)$; only the product of the two variables needs to be specified. We will use $[W(\vec{r})]$ to introduce normalization constraints which shall be obeyed at all times during a transient. Specifically, we define

$$T(t) = \langle [W]^T [v]^{-1} [\phi] \rangle$$

and it follow that, following (73), $[S(\vec{r}, t)]$ must obey the following constraint,

$$\langle [W]^T [v]^{-1} [S(\vec{r}, t)] \rangle = 1$$

where the symbol $\langle\ \rangle$ means spatial integration over the whole domain of the nuclear core. The factor $T(t)$ is called the amplitude function. Note that $[S(\vec{r}, t)]$ represents in some sense the total number of neutrons in the reactor, but that this number depends on the weight function. As the constraint on $[S(\vec{r}, t)]$ does not depend on time, the shape function may change in time, but its integral is time independent. Thus $T(t)$ itself represents the neutron population change in the reactor.

We can now obtain a differential equation for the time dependent variable $T(t)$ by replacing $[\phi(\vec{r}, t)]$ by the product $[S(\vec{r}, t)]T(t)$ in the space-time kinetics equations, by pre-multiplying the resulting equations by $[W(\vec{r})]^T$ and by integrating over the whole core volume. We thus obtain

$$\langle [W]^T [v]^{-1} [S] \rangle \frac{\partial T}{\partial t}$$

$$= \langle [W]^T \left\{ \begin{array}{c} \nabla \cdot [D] \vec{\nabla} [S] - [\Sigma][S] \\ + \left( (1-\beta)[\chi^p] + \sum_{i=1}^{D} \beta_i [\chi_i^d] \right) [\nu\Sigma_f]^T [S] \end{array} \right\} \rangle T(t)$$

$$- \langle [W]^T \sum_{i=1}^{D} \beta_i [\chi_i^d] [\nu\Sigma_f]^T [S] \rangle T(t)$$

$$+ \sum_{i=1}^{D} \lambda_i \langle [W]^T [\chi_i^d] C_i \rangle$$

where, in order to conform to certain conventions, we have added and subtracted the term in

$$\sum_{i=1}^{D} \beta_i [\chi_i^d] [\nu\Sigma_f]^T [S]$$

Very similar operations are performed on the precursor equations, that we pre-multiply by $[W]^T [\chi_i^d]$ before integrating over space to get

$$\frac{\partial}{\partial t} \langle [W]^T [\chi_i^d] C_i \rangle = \langle [W]^T [\chi_i^d] \beta_i [\nu\Sigma_f]^T [S] \rangle T - \lambda_i \langle [W]^T [\chi_i^d] C_i \rangle$$

We now *define* the following quantities

$$C_i(t) = \frac{\langle [W]^T [\chi_i^d] C_i \rangle}{\langle [W]^T [v]^{-1} [S] \rangle} \qquad \text{(EQ 74)}$$

$$\Lambda(t) = \frac{\langle [W]^T [v]^{-1} [S] \rangle}{\langle [W]^T \left\{ (1-\beta)[\chi^p] + \sum_{i=1}^{D} \beta_i [\chi_i^d] \right\} [\nu\Sigma_f]^T [S] \rangle} \quad \text{(EQ 75)}$$

$$\beta_i(t) = \beta_i \frac{\langle [W]^T [\chi_i^d] [\nu\Sigma_f]^T [S] \rangle}{\langle [W]^T \left\{ (1-\beta)[\chi^p] + \sum_{i=1}^{D} \beta_i [\chi_i^d] \right\} [\nu\Sigma_f]^T [S] \rangle} \quad \text{(EQ 76)}$$

$$\beta(t) = \sum_{i=1}^{D} \beta_i(t) \quad \text{(EQ 77)}$$

$$\rho = \frac{\left\langle [W]^T \left\{ \begin{array}{c} \nabla \cdot [D] \vec{\nabla} [S] - [\Sigma][S] \\ + \left( (1-\beta)[\chi^p] + \sum_{i=1}^{D} \beta_i [\chi_i^d] \right) [\nu\Sigma_f]^T [S] \end{array} \right\} \right\rangle}{\langle [W]^T \left\{ (1-\beta)[\chi^p] + \sum_{i=1}^{D} \beta_i [\chi_i^d] \right\} [\nu\Sigma_f]^T [S] \rangle} \quad \text{(EQ 78)}$$

With these definitions, the space-time kinetics equations become

$$\begin{aligned} \frac{d}{dt} T &= \frac{\rho - \beta}{\Lambda} T + \sum_{i=1}^{D} \lambda_i C_i \\ \frac{d}{dt} C_i &= \frac{\beta_i}{\Lambda} T - \lambda_i C_i \end{aligned} \quad \text{(EQ 79)}$$

which are the point kinetics equations.

## Point Kinetics Approximation

Up to now, there are no approximations in the point kinetics formulation. However, the parameters $\rho(t)$, $\beta_i(t)$ and $\Lambda(t)$ depend, by definition, on the shape function $[S(\vec{r}, t)]$.

Knowing $[S(\vec{r}, t)]$ means in turn knowing the neutron flux $[\phi(\vec{r}, t)]$ which necessitates a complete spatial solution. It then becomes very difficult to determine the point kinetics parameters.

The most common way of resolving this is to replace $[S(\vec{r}, t)]$ by a function depending only on space, denoted by $[S(\vec{r})]$. This function usually comes from the static solution of the reactor in the initial state before perturbations were applied to it. In this case, the parameters can only be approximate.

In effect, the $\beta_i$ and $\Lambda$ lose their time dependence. The reactivity depends on the temporal variations of the cross-sections and of the diffusion coefficients. But these are now applied to a shape function that is not representative of the actual state of the reactor during the transient.

In this case, it can be shown that the choice of the weight function $[W]$, which we have left arbitrary, can influence the precision of the solution of the point kinetics equations. In fact, the amplitude $T(t)$ depends very strongly on the reactivity $\rho(t)$. We wont show it here, but it can be proven that if the weight function is the adjoint flux of the initial static reactor, then the error in the reactivity estimation is much

reduced. It follows then that the error on the solution will be reduced too.

In spite of all these difficulties, point kinetics is still the most widely used method in kinetics. This comes from the small number of equations to solve, together with only one spatial (static, initial) calculation to do. However this apparent simplicity hides many difficulties.

## Analytic Solution

Finding analytic solutions to the point kinetics equations can teach much about the dynamics of a nuclear core. We examine this problem here.

Define a vector

$$[\psi] = \begin{bmatrix} T \\ C_1 \\ . \\ C_D \end{bmatrix}$$

and the following matrix

$$[R] = \begin{bmatrix} \frac{\rho-\beta}{\Lambda} & \lambda_1 & \lambda_2 & . & \lambda_D \\ \frac{\beta_1}{\Lambda} & -\lambda_1 & & & \\ \frac{\beta_2}{\Lambda} & & -\lambda_2 & & \\ . & & & . & \\ \frac{\beta_D}{\Lambda} & & & & -\lambda_D \end{bmatrix}$$

so that the point kinetics equations become

$$\frac{\partial}{\partial t}[\psi] = [R][\psi] \qquad \text{(EQ 80)}$$

In general, the reactivity varies in a complicated fashion in time. We chose a time interval during which reactivity is relatively constant. This is a step approximation of the function $\rho(t)$.

Reactivity will thus be constant in the time interval that we consider, and this simplifies the sought analytic solution.

We now introduce a vector $[\psi']$ related to the vector $[\psi]$ by the application of a linear transformation $[\Gamma]$,

$$[\psi] = [\Gamma][\psi']$$

System (80) then becomes,

$$[\Gamma]\frac{\partial}{\partial t}[\psi'] = [R][\Gamma][\psi']$$

which can be written

$$\frac{\partial}{\partial t}[\psi'] = [\Gamma]^{-1}[R][\Gamma][\psi'] \qquad \text{(EQ 81)}$$

where we have used the fact that in the time interval when reactivity is constant, so is $[\Gamma]$ a constant.

If the linear operator $[\Gamma]$ is chosen such that it diagonalizes the matrix in system (81), we have

$$\frac{\partial}{\partial t}[\psi'] = [\mathcal{D}][\psi'] \qquad \text{(EQ 82)}$$

where $[\mathcal{D}]$ is a diagonal matrix. The individual elements of $[\psi']$ are easy to calculate in this basis, becoming simply

$$[\psi'(t)] = \exp(\omega_i(t - t_0))[\psi'(t_0)]$$

Using the inverse of the linear transformation will give

$$[\Gamma]^{-1}[\psi(t)] = \exp(\omega_i(t - t_0))[\Gamma]^{-1}[\psi(t_0)]$$

which becomes

$$[\psi(t)] = [\Gamma]\exp(\omega_i(t - t_0))[\Gamma]^{-1}[\psi(t_0)]$$

We only have to find the elements of the $[\Gamma]$ matrix, as well as the $\omega_i$. To get them, we reconsider systems (81) and (82), which give

$$[\Gamma]^{-1}[R][\Gamma] = [\mathcal{D}]$$

A result of linear algebra[1] shows that the elements of the diagonal matrix $[\mathcal{D}]$ are the eigenvalues of the matrix $[R]$, and that the columns of the $[\Gamma]$ matrix are then made of the corresponding eigenvectors. Furthermore, the elements of the matrix $[\Gamma]^{-1}$ have as lines the eigenvectors of the (properly normalized) adjoint system. The problem is thus the same as calculating the eigenvalue-eigenvector problem of the matrix $[R]$.

To calculate the eigenvalues, we only have to find the values of $\omega_i$ that will make the determinant of the matrix $[R - \omega_i I]$ equal to zero, which gives rise to the equation

$$\det\left([R - \omega_i I]\right) = 0$$

Let us expand this determinant along the first column of the matrix $[R - \omega_i I]$. We find easily that

$$\left\{\left(\frac{\rho - \beta}{\Lambda} - \omega\right) + \sum_{i=1}^{D} \frac{\beta_i \lambda_i}{\Lambda(\lambda_i + \omega)}\right\} \prod_{j=1}^{D} (-\lambda_i - \omega) = 0$$

1. see for example J. H. Wilkinson, The Algebraic Eigenvalue Problem, Oxford University Press, Oxford, 1965.

If we remark that $\omega = -\lambda_i$ is not a root of this expression, then the $\omega$ must necessarily be the solutions of

$$\left\{\left(\frac{\rho-\beta}{\Lambda}-\omega\right)+\sum_{i=1}^{D}\frac{\beta_i\lambda_i}{\Lambda(\lambda_i+\omega)}\right\}\prod_{j=1}^{D}(-\lambda_i-\omega)=0$$

or, after a slight rearrangement of the various terms,

$$\rho=\omega\Lambda+\sum_{i=1}^{D}\frac{\beta_i\omega}{\lambda_i+\omega} \qquad \text{(EQ 83)}$$

This is Nordheim's equation, whose zeroes can only be found approximately (Newton's method, etc.) when more than three delayed neutron families are at play.

After calculating the eigenvalues, we have to calculate the eigenvectors, which we denote by

$$[U]^i=\begin{bmatrix}u_0^i\\u_1^i\\u_2^i\\.\\u_D^i\end{bmatrix}$$

where the index i indicates that the eigenvector belongs to the eigenvalue $\omega_i$.

In this type of calculation, there is always an element of the eigenvector that can be chosen arbitrarily. We take as first element of the eigenvector a value of 1. With such a choice, we will have

$$\left[U\right]^i = \begin{bmatrix} 1 \\ u_1^i \\ u_2^i \\ . \\ u_D^i \end{bmatrix}$$

In order to find the other components, we only have to solve the system

$$\left[R - \omega_i I\right]\left[U\right]^i = 0$$

or, in more direct terms,

$$\begin{bmatrix} \frac{\rho-\beta}{\Lambda} - \omega_i & \lambda_1 & \lambda_2 & . & \lambda_D \\ \frac{\beta_1}{\Lambda} & -\lambda_1 - \omega_i & & & \\ \frac{\beta_2}{\Lambda} & & -\lambda_2 - \omega_i & & \\ . & & & . & \\ \frac{\beta_D}{\Lambda} & & & & -\lambda_D - \omega_i \end{bmatrix} \begin{bmatrix} 1 \\ u_1^i \\ u_2^i \\ . \\ u_D^i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ . \\ 0 \end{bmatrix}$$

We then get

$$u_j^i = \frac{\beta_j}{\Lambda(\lambda_j + \omega_i)}$$

and it becomes possible to construct the matrix $[\Gamma]$,

$$[\Gamma] = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \frac{\beta_1}{\Lambda(\lambda_1 + \omega_0)} & \frac{\beta_1}{\Lambda(\lambda_1 + \omega_1)} & \cdots & \frac{\beta_1}{\Lambda(\lambda_1 + \omega_D)} \\ \frac{\beta_2}{\Lambda(\lambda_2 + \omega_0)} & \frac{\beta_2}{\Lambda(\lambda_2 + \omega_1)} & \cdots & \frac{\beta_2}{\Lambda(\lambda_2 + \omega_D)} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \frac{\beta_D}{\Lambda(\lambda_D + \omega_0)} & \frac{\beta_D}{\Lambda(\lambda_D + \omega_1)} & \cdots & \frac{\beta_D}{\Lambda(\lambda_D + \omega_D)} \end{bmatrix}$$

The matrix $[\Gamma]^{-1}$ can be obtained by direct inversion of $[\Gamma]$ or by solving the adjoint problem, which is much easier than direct inversion

$$[R - \omega_i I]^T [U^*]^i = 0$$

We then get that

$$\left[\Gamma\right] \propto \begin{bmatrix} 1 & \frac{\lambda_1}{\lambda_1+\omega_0} & \frac{\lambda_2}{\lambda_2+\omega_0} & \cdots & \frac{\lambda_D}{\lambda_D+\omega_0} \\ 1 & \frac{\lambda_1}{\lambda_1+\omega_1} & \frac{\lambda_2}{\lambda_2+\omega_1} & \cdots & \frac{\lambda_D}{\lambda_D+\omega_1} \\ . & . & . & . & . \\ . & . & . & . & . \\ 1 & \frac{\lambda_1}{\lambda_1+\omega_D} & \frac{\lambda_2}{\lambda_2+\omega_D} & \cdots & \frac{\lambda_D}{\lambda_D+\omega_D} \end{bmatrix}$$

The proportionality sign is used to emphasize that the lines of the matrix has not been normalized with the columns of $\left[\Gamma\right]$. This final step is left to the reader.

The determination of the $\omega_i$ shows that they differ a lot one from the other, the minimum being about $\frac{\rho-\beta}{\Lambda}$, whereas the maximum is greater than $-\lambda_1$ in all cases, and greater than 0 if $\rho > 0$. Such a large spread in eigenvalues will cause difficulties to the numerical methods used for solving the point kinetics equations. The system of equations is a stiff system.

## Conclusion

We can then conclude on the following observations:

1. With more than one group of delayed neutrons, the analytic solution of the point kinetics equations becomes difficult to get.

2. It is impossible to find a true analytic solution, because Nordheim's equation (83) is transcendental in the $\omega_i$, which can then only be obtained through numerical methods.
3. The eigenvectors of the adjoint problem must be calculated, or the matrix $[\Gamma]$ must be directly inverted.
4. As the reactivity changes in time, all this work has to be done at each time interval

Because of all this, the analytic solution of the point kinetics equations constitutes a method which is too costly and difficult to be considered practical.