Material derivative

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In mathematics, the **material derivative**[1][2] is a derivative taken along a path moving with velocity **v**, and is often used in fluid mechanics and classical mechanics. It describes the time rate of change of some quantity (such as heat or momentum) by following it, while moving with a — space and time dependent — velocity field.

For example, in fluid dynamics, take the case that the velocity field under consideration is the flow velocity itself, and the quantity of interest is the temperature of the fluid. Then the material derivative describes the temperature evolution of a certain fluid parcel in time, as it is being moved along its pathline (trajectory) while following the fluid flow.

There are many other names for this operator, including:

- **convective derivative**[3]
- **advective derivative**
- **substantive derivative**[4]
- **substantial derivative**[1]
- **Lagrangian derivative**[5]
- **Stokes derivative**[4]
- **particle derivative**
- **hydrodynamic derivative**[1]
- **derivative following the motion**[1]

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Definition

The material derivative of a scalar field $\varphi(\mathbf{x}, t)$ and a vector field $\mathbf{u}(\mathbf{x}, t)$ is defined respectively as:

$$\frac{D\varphi}{Dt} = \frac{\partial\varphi}{\partial t} + \mathbf{v}\cdot\nabla\varphi,$$

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial\mathbf{u}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{u},$$

where the distinction is that $\nabla\varphi$ is the gradient of a scalar, while $\nabla\mathbf{u}$ is the tensor derivative of a vector. In case of the material derivative of a vector field, the term $\mathbf{v}\bullet\nabla\mathbf{u}$ can both be interpreted as $\mathbf{v}\bullet(\nabla\mathbf{u})$ involving the tensor derivative of **u**, or as $(\mathbf{v}\bullet\nabla)\mathbf{u}$, leading to the same result.[6]

Confusingly, the term convective derivative is both used for the whole material derivative $D\varphi/Dt$ or $D\mathbf{u}/Dt$, and for only the spatial rate of change part , $\mathbf{v}\bullet\nabla\varphi$ or $\mathbf{v}\bullet\nabla\mathbf{u}$ respectively.[2] For that case, the convective derivative only equals D/Dt for time independent flows.

These derivatives are physical in nature and describe the transport of a scalar or vector quantity in a velocity field $\mathbf{v}(\mathbf{x}, t)$. The effect of the time independent terms in the definitions are for the scalar and vector case respectively known as advection and convection.

It may be shown that, in orthogonal coordinates, the j^{th} component of convection is given by[7]:

$$[\mathbf{v}\cdot\nabla\mathbf{u}]_j = \sum_i \frac{v_i}{h_i}\frac{\partial u_j}{\partial q^i} + \frac{u_i}{h_i h_j}\left(v_j \frac{\partial h_j}{\partial q^i} - v_i \frac{\partial h_i}{\partial q^j}\right).$$

Development

Consider a scalar quantity $\varphi = \varphi(\mathbf{x}, t)$, where t is understood as time and **x** as position. This may be some physical variable such as temperature or chemical concentration. The physical quantity exists in a fluid, whose velocity is represented be the vector field $\mathbf{v}(\mathbf{x}, t)$.

The (total) derivative with respect to time of φ is expanded through the multivariate chain rule:

$$\frac{d}{dt}(\varphi(\mathbf{x},t)) = \frac{\partial\varphi}{\partial t} + \nabla\varphi\cdot\frac{d\mathbf{x}}{dt}$$

It is apparent that this derivative is dependent on the vector

$$\frac{d\mathbf{x}}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$$

which describes a *chosen* path $\mathbf{x}(t)$ in space. For example, if $d\mathbf{x}/dt = 0$ is chosen, the time derivative becomes equal to the partial derivative, which agrees with the definition of a partial derivative: a derivative taken with respect to some variable (time in this case) holding other variables constant (space in this case). This makes sense because if $d\mathbf{x}/dt = 0$, then the derivative is taken as some *constant* position. This static position derivative is called the Eulerian derivative.

An example of this case is a swimmer standing still and sensing temperature change in a lake early in the morning: the water gradually becomes warmer due to heating from the sun.

If, instead, the path $\mathbf{x}(t)$ is not a standstill, the (total) time derivative of φ may change due to the path. For example, imagine the swimmer is in a motionless pool of water, indoors and unaffected by the sun. One end happens to be a constant hot temperature and the other end a constant cold temperature, by swimming from one end to the other the swimmer senses a change of temperature with respect to time, even though the temperature at any given (static) point is a constant. This is because the derivative is taken at the swimmer's changing location. A temperature sensor attached to the swimmer would show temperature varying in time, even though the pool is held at a steady temperature distribution.

The material derivative finally is obtained when the path $\mathbf{x}(t)$ is chosen to have a velocity equal to the fluid velocity:

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}$$

That is, the path follows the fluid current described by the fluid's velocity field **v**. So, the material derivative of the scalar φ is:

An example of this case is a lightweight, neutrally buoyant particle swept around in a flowing river undergoing temperature changes, maybe due to one portion of the river being sunny and the other in a shadow. The water as a whole may be heating as the day progresses. The changes due to the particle's motion (itself caused by fluid motion) is called *advection* (or convection if a vector is being transported).

The definition above relied on the physical nature of fluid current; however no laws of physics were invoked (for example, it hasn't been shown that a lightweight particle in a river will follow the velocity of the water). It turns out, however, that many physical concepts can be written concisely with the material derivative. The general case of advection, however, relies on conservation of mass in the fluid stream; the situation becomes slightly different if advection happens in a non-conservative medium.

Only a path was considered for the scalar above. For a vector, the gradient becomes a tensor derivative; for tensor fields we may want to take into account not only translation of the coordinate system due to the fluid movement but also its rotation and stretching. This is achieved by the upper convected time derivative.

See also

- Navier–Stokes equations
- Euler equations

References

Inline

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General

- Structure and Interpretation of Classical Mechanics (http://mitpress.mit.edu/SICM/book-Z-H-13.html#%_sec_Temp_122)
- Fluid Mechanics by Kundu and Cohen, 3rd Edition
- Introduction to Continuum Mechanics by Lai, Rubin, and Krempl, 3rd edition

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