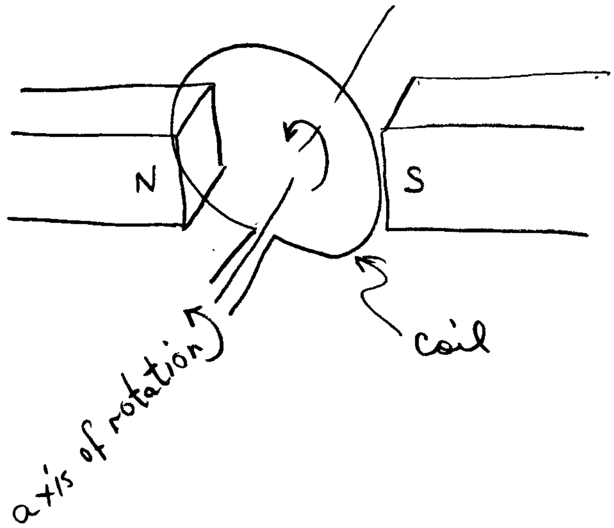
Chapter 33 Alternating Current Circuits

33.1 AC Sources and Phasors



When a coil is rotated in a magnetic field, an Emf is induced proportional to the change in the number of B field lines crossing through the loop.
(see Chapter 32 for details)

Thus, a coil rotating at an angular frequency, ω rad/s, generates a voltage $v = V_{max} \sin \omega t$

where $\omega = 2\pi f \leftarrow$ f = frequency, cycles/s

$= \dfrac{2\pi}{T} \leftarrow$ T = period, s.

In North America, $f = 60$ cycles/s $= 60$ Hz (Hertz)

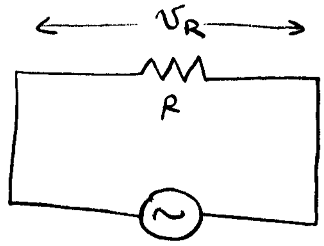
$\therefore \omega = 2\pi \times 60 = 377$ rad/s.

This chapter uses <u>phasor</u> diagrams to investigate ac circuit behaviour involving:

resistors (R)
inductors (L)
capacitors (C)

Applications: household and industrial wiring.

33.2 Resistors in an AC Circuit



$v = V_{max} \sin \omega t$

(Note the symbol for an ac generator)

From Kirchhoff's loop equation:

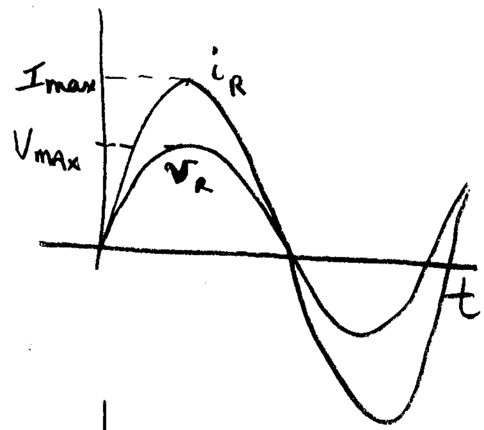
$\sum \Delta V = 0$

$\therefore \; v - v_R = 0$

$\therefore \; v = v_R = V_{max} \sin \omega t$ = instaneous voltage drop across R

$\therefore \; i_R = \dfrac{v}{R} = \dfrac{V_{max}}{R} \sin \omega t = I_{max} \sin \omega t$

Thus i_R is <u>in phase</u> with v_R



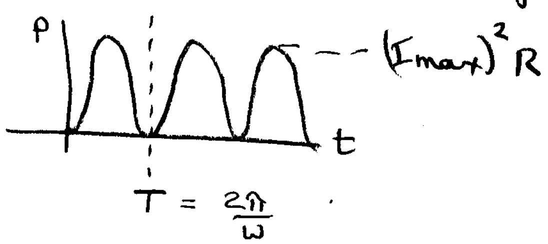
<u>Notes</u>:

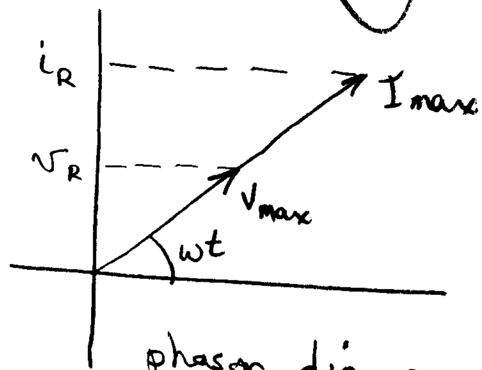
Average of i_R or $v_R = 0$

Power dissipated in the resistor:

$P = i^2 R$

$P_{average} = \int_0^T P(t)\, dt = (i^2)_{average} R$





phasor diagram

$R I_{max}^2 \int_0^T \sin^2 \omega t \, dt = \dfrac{I_{max}^2 R}{2}$

$I_{rms} = \sqrt{\dfrac{I_{max}^2}{2}} = \dfrac{1}{\sqrt{2}} I_{max} = 0.707 \, I_{max}$

root mean square $\equiv \sqrt{\text{average}(i^2)}$

<u>Note</u>: DC power at current I_{rms} is same as AC power with current $I_{max} \sin(\omega t)$

Similarily, $V_{rms} = \frac{V_{max}}{\sqrt{2}} = 0.707\, V_{max}$

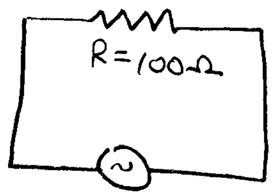
The 120 V household current is really a V_{rms} of 120 V.

Thus $V_{max} = \frac{120}{0.707} = 170$ V.

Multimeters typically display rms values.

<u>Example</u> <u>rms Current</u>

Find V_{rms}, I_{rms}



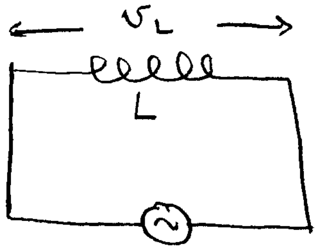
$v = 200 \sin \omega t$

$V_{rms} = \frac{V_{max}}{\sqrt{2}} = \frac{200}{\sqrt{2}} = \underline{141\text{ V}}$

$I_{rms} = \frac{V_{rms}}{R} = \frac{141\text{ V}}{100\ \Omega} = \underline{1.41\text{ A}}$

33.3 Inductors in an AC Circuit

Using Kirchhoff again:

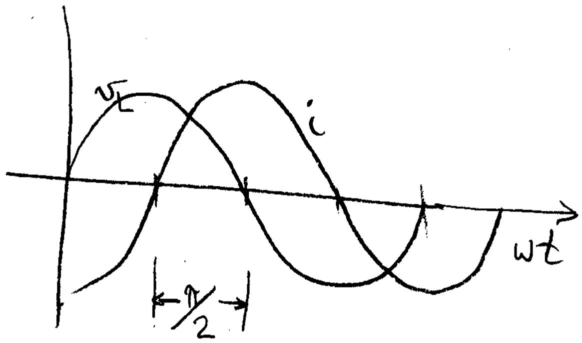


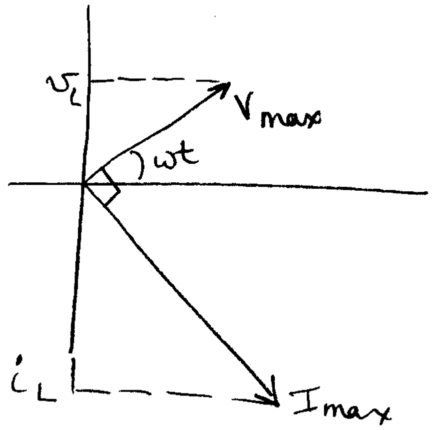
$$v_L - L\frac{di}{dt} = 0$$

or $$L\frac{di}{dt} = V_{max} \sin \omega t$$

Integrating, $$i_L = -\frac{V_{max}}{\omega L} \cos \omega t = \frac{V_{max}}{\omega L} \sin(\omega t - \pi/2)$$

ie current in an inductor <u>lags</u> voltage by 90°.





Define inductive reactance:

$$X_L = \omega L$$

$$\therefore \quad I_{max} = \frac{V_{max}}{\omega L} = \frac{V_{max}}{X_L}$$

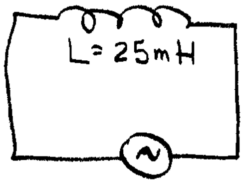
X_L ← units of Ω.
~ to resistor

$X_L \uparrow$ as $\omega \uparrow$

Makes physical sense – inductors have "inertia" like mass.

Example A Purely Inductive AC Circuit

Find X_L and I_{rms}.



⇐ H = Henries

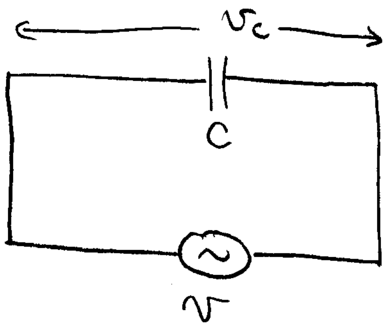
$V_{rms} = 150\,v$

$w = 60\,Hz$

$$X_L = \omega L = (2\pi \times 60)\,s^{-1}\,(0.025)\,H$$
$$= \underline{9.42\,\Omega}.$$

$$I_{rms} = \frac{V_{rms}}{X_L} = \frac{150}{9.42} = \underline{15.9\ A}$$

33.4 Capacitors in an AC Circuit



$v - v_C = 0$

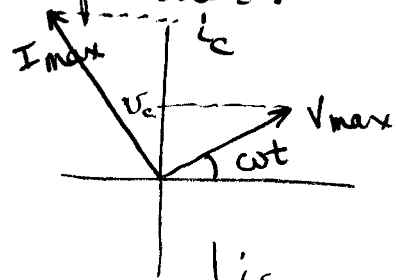
$\therefore\ v = v_C = V_{max} \sin \omega t$

But $v_C = \dfrac{Q}{C}$

$\therefore\quad Q = C V_{max} \sin \omega t$

$\therefore\ i = \dfrac{dQ}{dt} = \omega C V_{max} \cos \omega t = \omega C V_{max} \sin(\omega t + \pi/2)$

Thus current **leads** voltage in a capacitor.

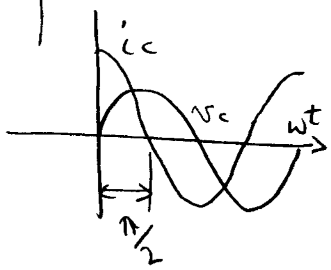


Define capacitive reactance:

$X_C = \dfrac{1}{\omega C}$

$\therefore\quad I_{max} = \dfrac{V_{max}}{X_C}$ ← units of Ω.

$\sim$ to resistance.

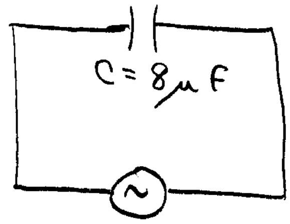


$X_C \uparrow$ as $\omega \downarrow$

Makes physical sense - capacitor acts like a storage unit. Tends to an open circuit as you approach dc ($\omega = 0$, $X_C = \infty \Rightarrow I_{max} = 0$).

Example: <u>A Purely Capacitive AC Circuit</u>

Find X_c, I_{rms}.



$v_{rms} = 150$ V

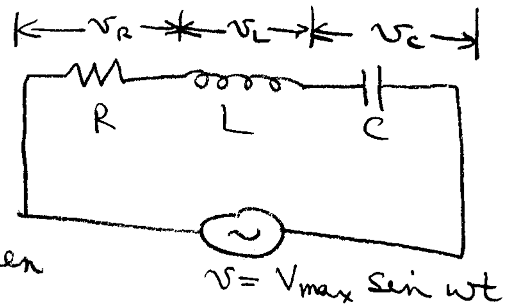
$\omega = 60$ Hz

$$X_c = \frac{1}{\omega C} = \frac{1}{(2\pi \times 60)(8 \times 10^{-6})}\ \Omega$$

$$= \underline{332\ \Omega}$$

$$I_{rms} = \frac{150}{332} = \underline{0.452\ A.}$$

33.5 The RLC Series Circuit

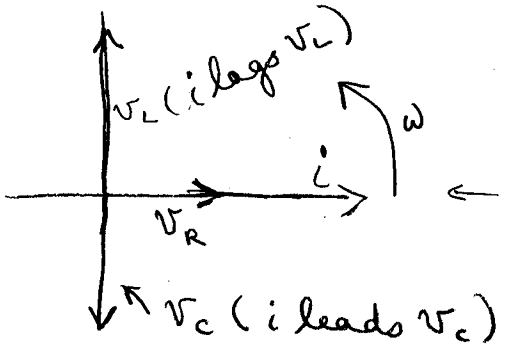


Let us assume that

$$i = I_{max}\sin(\omega t - \phi)$$

where ϕ is the phase angle between $i + v$.

The current is continuous + the same everywhere in the loop. The component voltages are as shown:

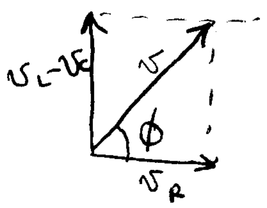


$$\therefore\ v = v_R + v_L + v_C$$

$$= I_{max}R\sin\omega t + I_{max}X_L\underbrace{\sin(\omega t + \pi/2)}_{=\cos\omega t} + I_{max}X_C\underbrace{\sin(\omega t - \pi/2)}_{=-\cos\omega t}$$

$$= V_R\sin\omega t + V_L\sin(\omega t + \pi/2) + V_C\sin(\omega t - \pi/2)$$

Take a vector sum:



$$\therefore\ V_{max} = \sqrt{V_R^2 + (V_L - V_C)^2}$$

$$= I_{max}\sqrt{R^2 + (X_L - X_C)^2}$$

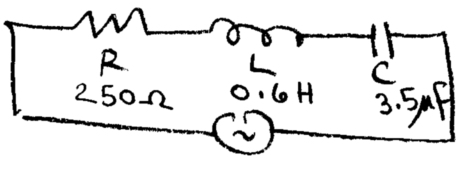
$$\equiv I_{max}\,Z$$

↑ impedence

$$\tan\phi = \frac{X_L - X_C}{X_R}$$

Example: Analyzing a Series RLC Circuit

Find Z, ϕ, v_R, v_L, v_C



$v = 150 \sin \omega t$
$f = 60\ Hz$

$\omega = 2\pi f = 377\ sec^{-1}$

$\therefore X_L = \omega L = 266\ \Omega$

$+ X_C = \frac{1}{\omega C} = 758\ \Omega$

$\therefore Z = \sqrt{250^2 + (226 - 758)^2}$

$= 588\ \Omega$

$I_{max} = \frac{V_{max}}{Z} = \frac{150}{588} = 0.255\ A$

$V_R = I_{max} R = 0.255 \times 250$
$= 63.8\ V$

$V_L = I_{max} X_L = 57.6\ V$

$V_C = I_{max} X_C = 193\ V$

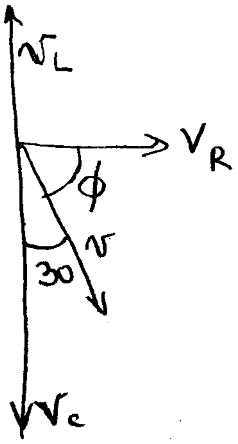
$\phi = \tan^{-1}\left(\frac{X_L - X_C}{X_R}\right) = \tan^{-1}\left(\frac{226 - 758}{250}\right) = -64.8°$

ie $\phi < 0$, $\therefore$ current leads the voltage
(more capacitance than inductance)

$\therefore v = 63.8 \sin 377t + 57.6 \cos 377t - 193 \cos 377t$

Example: Finding L from a Phasor Diagram

Given $R = 800\,\Omega$, $C = 4\,\mu F$, $f = 60\,Hz$, v_c lags v by $30°$.
Find L in a series RLC circuit.



We have: $\tan\phi = \dfrac{X_L - X_C}{X_R}$

$\therefore X_L = X_C + X_R \tan\phi$

$X_C = \dfrac{1}{377 \times 4 \times 10^{-6}}\,\Omega = 663.1\,\Omega$

$X_R = 800\,\Omega$

$\tan\phi = \tan 60° = 1.732$

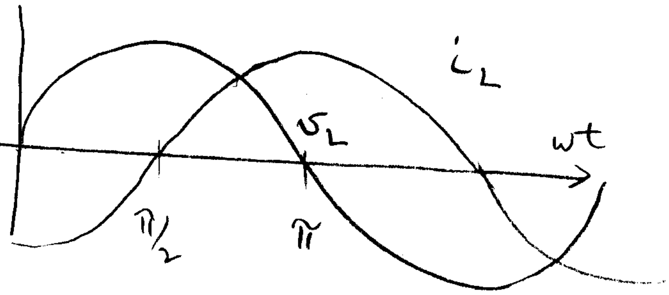
$\therefore X_L = 2048\,\Omega \quad \Rightarrow L = \dfrac{X_L}{\omega} = \underline{\underline{5.44\,H}}$

33.6 Power in an AC Circuit

Instantaneous power in an inductor $= i_L v_L$

$$= \frac{V_{max}}{\omega L} \sin(\omega t - \pi/2)\, V_{max} \sin\omega t = \frac{-V_{max}^2}{\omega L} \sin\omega t \cos\omega t$$

Starting from $t=0$, the instantaneous power, $i_L v_L$, changes sign every 1/4 cycle (ie every $\pi/2$). Thus, the maximum stored energy is $\int_{\pi/2\omega}^{\pi/\omega} i_L(t) v_L(t)\,dt = \frac{1}{\omega}\int_{\pi/2}^{\pi} i_L(\theta) v_L(\theta)\,d\theta$



$(\theta = \omega t)$

$$= -\frac{V_{max}^2}{\omega^2 L}\int_{\pi/2}^{\pi} \sin\theta \cos\theta\, d\theta = -\frac{V_{max}^2}{\omega^2 L}\int_{\pi/2}^{\pi} \underset{u}{\sin\theta}\, \underbrace{d(\sin\theta)}_{du}$$

$$= \frac{-V_{max}^2}{\omega^2 L}\left[\frac{\sin^2\theta}{2}\right]_{\pi/2}^{\pi} = +\frac{V_{max}^2}{2\omega^2 L} = \frac{1}{2\omega} I_{max} V_{max}$$

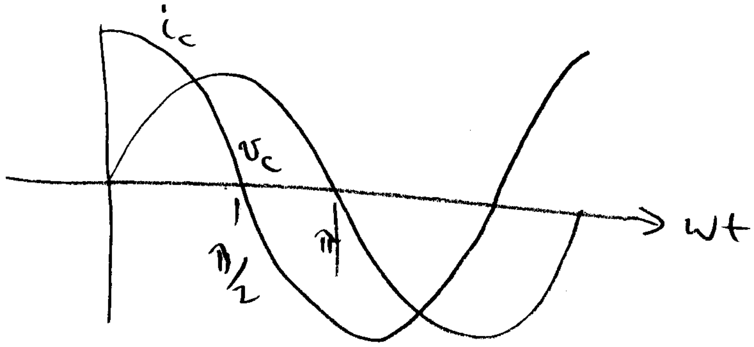
$$= \frac{1}{2} L I_{max}^2 \quad \text{since} \quad V_{max} = I_{max}\,\omega L$$

But this energy is given up to the circuit on the next 1/4 cycle. Thus, energy is not dissipated in an inductor.

Instantaneous power in a capacitor $= i_c v_c$

$$= V_{max} \cdot \omega C \cdot \sin(\omega t + \pi/2)\, V_{max} \sin \omega t$$

$$= V_{max}^2 (\omega C) \sin \omega t \cos \omega t$$



As per the inductor, the stored energy is

$$\int_0^{\pi/2\omega} i_c(t)\, v_c(t)\, dt = \frac{1}{\omega} \int_0^{\pi/2} i_c(\theta)\, v_c(\theta)\, d\theta$$

$$= V_{max}^2 \cdot \frac{\omega C}{\omega} \int_0^{\pi/2} \sin\theta \cos\theta \, d\theta$$

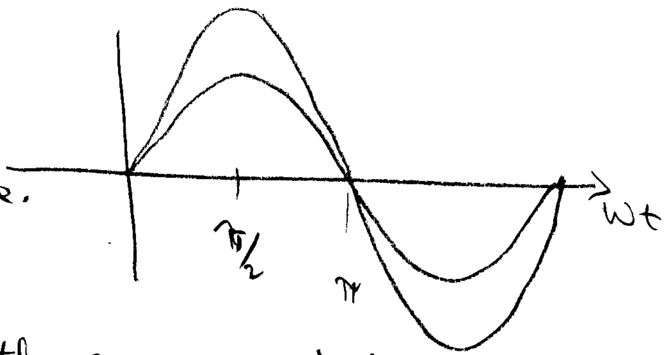
$$= \frac{C V_{max}^2}{2}$$

Again, this energy is given up on the next 1/4 cycle. Thus, energy is not dissipated in a capacitor.

Instantaneous power in a resistor $= i_R V_R$

$$= \frac{V_{max}}{R} \sin \omega t \cdot V_{max} \sin \omega t$$

$$= \frac{V_{max}^2}{R} \sin^2 \omega t$$



Note that power is always +ve.
(ie energy is dissipated)

In one complete cycle, the energy dissipated is

$$\int_0^{2\pi/\omega} i_R(t)\, v_R(t)\, dt = \frac{V_{max}^2}{\omega R} \int_0^{2\pi} \sin^2 \underbrace{\omega t}_{\theta}\, d\underbrace{(\omega t)}_{\theta}$$

$$= \frac{V_{max}^2}{\omega R} \cdot \pi \qquad \left[\text{Note } \int \sin^2 x\, dx = \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x \right]$$

The time for 1 cycle is $T = \frac{2\pi}{\omega}$

∴ The average power is $\left(\frac{V_{max}^2 \cdot \pi}{\omega R} \right) \Big/ T$

$$= \frac{1}{2} \frac{V_{max}^2}{R} = \frac{1}{2} I_{max}^2 R$$

Power in a RLC series circuit is, in general:

$$P(t) = i(t)\,v(t) = I_{max}\sin(\omega t - \phi)\, V_{max}\sin\omega t$$
$$= I_{max}V_{max}\sin\omega t\,\sin(\omega t - \phi)$$

Using the trig identity $\sin(a \pm b) = \sin a\cos b \pm \cos a \sin b$,

$$P(t) = I_{max}V_{max}\sin^2\omega t\cos\phi - I_{max}V_{max}\sin\omega t\cos\omega t\cos\phi$$

Integrating over 1 cycle:

$$P_{average} = \frac{1}{T}\int_0^T P(t)\,dt = \frac{1}{(2\pi/\omega)}\cdot\left(\frac{1}{\omega}\right)\int_0^{2\pi} P(\theta)\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} P(\theta)\,d\theta = \frac{I_{max}V_{max}\cos\phi}{2\pi}\left[\int_0^{2\pi}\sin^2\theta\,d\theta - \int_0^{2\pi}\sin\theta\cos\theta\,d\theta\right]$$

$$= \frac{I_{max}V_{max}\cos\phi}{2\pi}\left[\pi - 0\right] = \frac{I_{max}V_{max}\cos\phi}{2}$$

$$= I_{rms}V_{rms}\underbrace{\cos\phi}$$

this is called the power factor

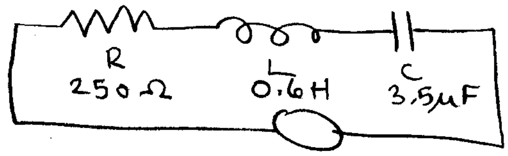
Note ~~that~~ since $\cos\phi = \dfrac{V_R}{V_{max}} = \dfrac{I_{max}R}{V_{max}}$ we have:

$$P_{average} = \frac{I_{max}\cancel{V_{max}}}{2}\cdot\frac{I_{max}R}{\cancel{V_{max}}} = \frac{I_{max}^2 R}{2} = I_{rms}^2 R$$

This is just what we found for the power dissipated in the resistor. Thus we are consistent.

Example: Average Power in a RLC Series Circuit

Find $P_{average}$.



$V = 150 \sin \omega t$
$f = 60\,\text{Hz}$

$$V_{rms} = \frac{V_{max}}{\sqrt{2}} = \frac{150}{\sqrt{2}} = 106\text{ V}$$

$$I_{rms} = \frac{I_{max}}{\sqrt{2}} = \frac{V_{max}/Z}{\sqrt{2}} = \frac{0.255\text{A}}{\sqrt{2}} = 0.180\text{A}$$

see example in 33.5 for calculation

$$\phi = -64.8°$$

$$\therefore\ P_{average} = I_{rms} V_{rms} \cos\phi = 0.180 \times 106 \times 0.426$$
$$= 8.13\text{ W}$$

compare to $P_{average}$ of resistor: $I_{rms}^2 R = (0.180)^2 \times 250\text{ W}$
$$= 8.13\text{ W.}$$

33.7 Resonance in a Series RLC Circuit

From 33.5 we found:

$$I_{max} = \frac{V_{max}}{Z} \quad \text{or equally} \quad I_{rms} = \frac{V_{Rms}}{Z}.$$

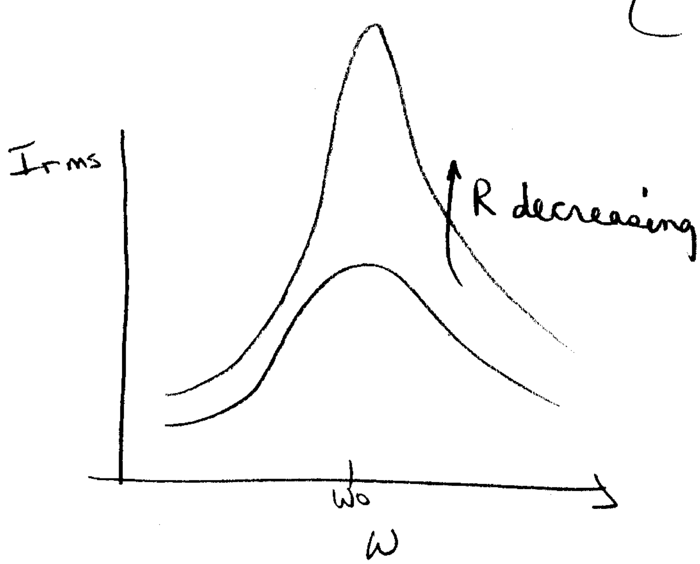
Thus

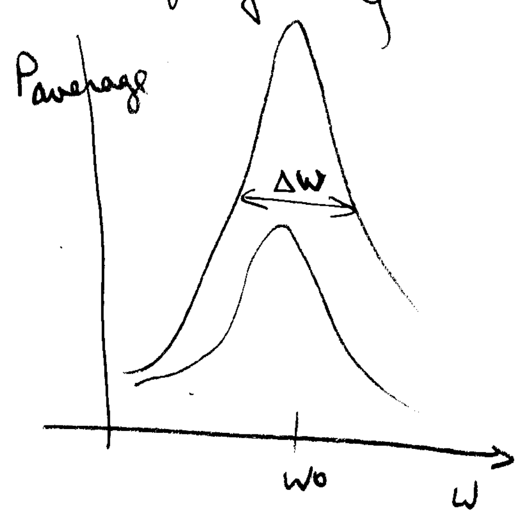
$$I_{rms} = \frac{V_{rms}}{\sqrt{R^2 + (X_L - X_C)^2}} \qquad \text{where } X_L = \omega L, \quad X_C = \frac{1}{\omega C}$$

Thus, if the parameters are varied (perhaps by varying ω), the denominator will change. The maximum I (minimum Z) occurs when $X_L = X_C$,

ie $\omega L = \frac{1}{\omega C} \Rightarrow \omega_0 = \frac{1}{\sqrt{LC}}$.

resonance frequency





$\Delta\omega$ = full width at ½ maximum.

Define $Q_0 = \frac{\omega}{\Delta\omega}$ = quality factor.

In terms of ω_0:

$$P_{average} = I_{rms}^2 R = \frac{V_{rms}^2 R}{\left(R^2 + (X_L - X_c)^2\right)}$$

$$= \frac{V_{rms}^2 R}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} = \frac{V_{rms}^2 R}{R^2 + \left(\frac{L}{\omega}\right)^2 \left(\omega^2 - \frac{1}{LC}\right)^2}$$

$$= \frac{V_{rms}^2 R}{R^2 + \left(\frac{L}{\omega}\right)^2 \left(\omega^2 - \omega_0^2\right)^2}$$

Get max. power when $\omega = \omega_0$

High $Q_0 \Leftrightarrow$ narrow range
Low $Q_0 \Leftrightarrow$ wide range
ie a circuit will tend to resonate only over a narrow frequency range if Q_0 is high.

<u>Example</u>: radio tuner

Mechanical systems resonate in a similar fashion.

Example: A Resonating RLC circuit

$R = 150\,\Omega$
$L = 20\text{ mH}$
$V_{rms} = 20\text{ V}$
$\omega = 5000\text{ sec}^{-1}$

What is C for peak current at ω?
What is the peak current?

$$\omega_0 = 5000\text{ sec}^{-1} = \frac{1}{\sqrt{LC}}$$

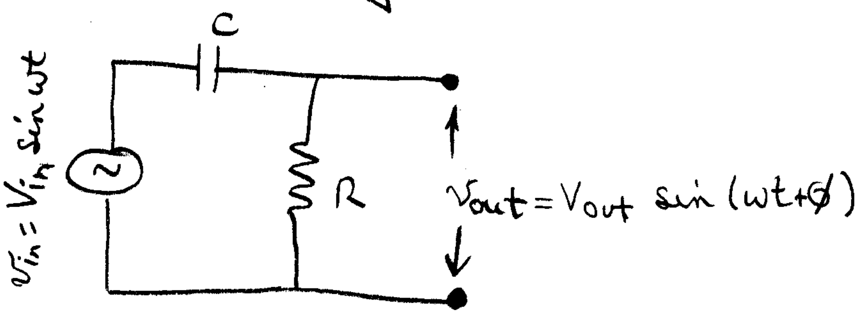
$$\therefore\ C = \left(\frac{1}{\omega_0^2 L}\right) = \frac{1}{(5000)^2 \times 20\times10^{-3}} = \underline{2\,\mu F}$$

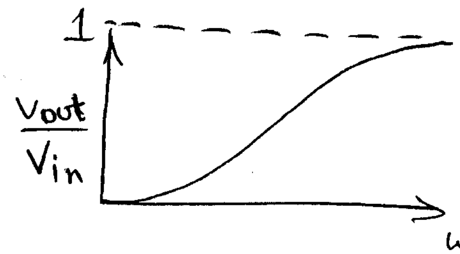
$$I_{max} = \frac{20\text{ V}}{\sqrt{(150)^2 + \underbrace{(X_L - X_C)^2}_{\approx 0}}} = \frac{20}{150} = \underline{0.133\text{ A}}$$

33.8 Filter Circuits

Used to filter out unwanted frequencies.

High-pass filter
(passes high freq.,
filters low freq.)





From our results on RLC series circuits,

$$V_{in} = I_{max} Z = I_{max}\sqrt{R^2 + \left(\tfrac{1}{\omega C}\right)^2}$$

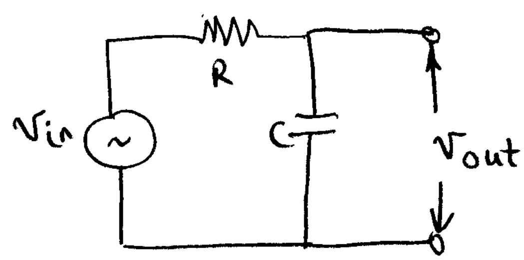
$$V_{out} = I_{max} R$$

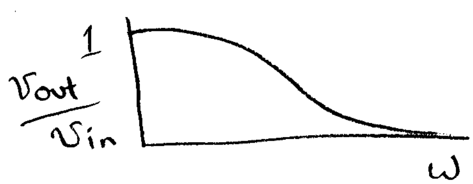
$$\therefore \frac{V_{out}}{V_{in}} = \frac{R}{\sqrt{R^2 + (1/\omega C)^2}} \rightarrow 0 \text{ as } \omega \rightarrow 0$$

Similarly, for a low-pass filter,

$$\frac{V_{out}}{V_{in}} = \frac{1/\omega C}{\sqrt{R^2 + (1/\omega C)^2}}$$

$\rightarrow 0$ as $\omega \rightarrow \infty$





33.9 The Transformer and Power Transmission

Because of i^2R losses, it is desirable to keep i low in transmission lines. Since $P = iV$, the voltage must, therefore, be high.

At the user end, we need low voltage and high current for safety and efficiency.

Thus we need transformers.

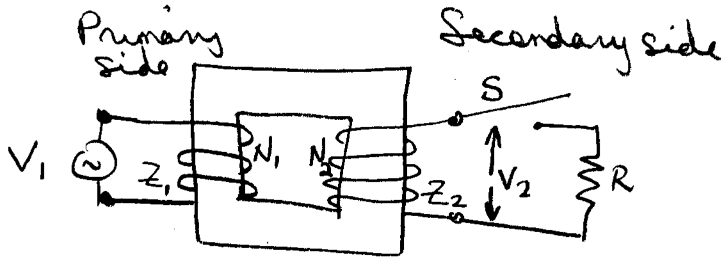
Transformer

Faraday's law says $V = -N \dfrac{d\Phi}{dt} = -N \dfrac{d(B \cdot A)}{dt}$

where Φ = magnetic flux = magnetic field × area of loop,

A changing B field or area, ie a changing flux, induces a voltage (Emf).

We use wire loops to create a B field (solenoid) and an iron core to contain and direct the field:



$$V_1 = -N_1 \frac{d\Phi}{dt}, \quad V_2 = -N_2 \frac{d\Phi}{dt} \Rightarrow V_2 = \frac{N_2}{N_1} V_1$$

Thus, we can easily make a step-up or a step-down transformer by varying N_1 + N_2.

If we close the switch, S, the induced voltage, V_2, induces a current I_2.

Since power in = power out (assuming no losses):

$$I_1 V_1 = I_2 V_2$$

Now $I_2 = V_2/R$ & $I_1 = V_1/R_{eq}$

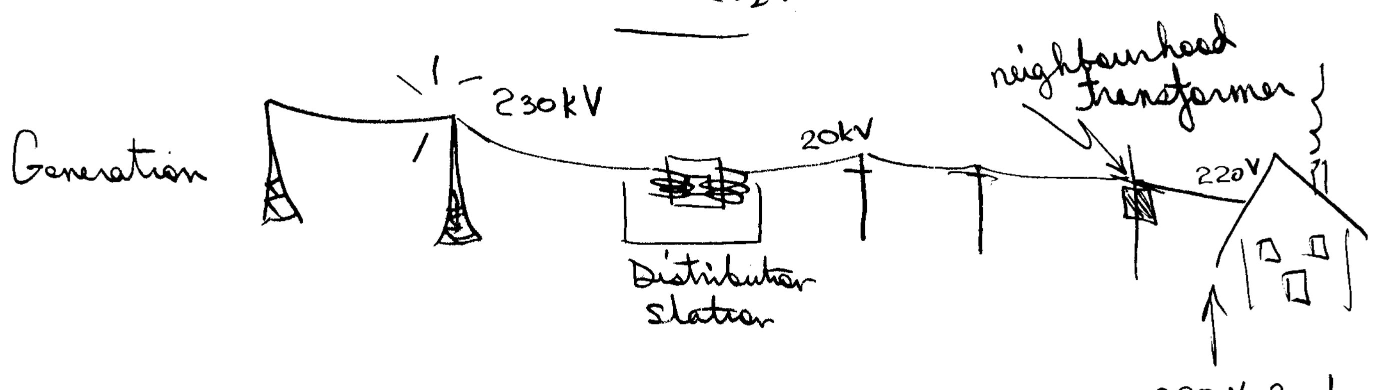
where Req is the equivalent R as seen from the primary side (ie side 1).

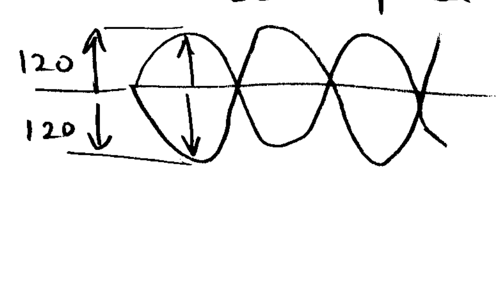
Since $V_1 = \frac{N_1}{N_2} V_2$, $I_1 = \frac{N_1}{N_2}\frac{V_2}{R_{eq}} = \frac{N_1}{N_2}\frac{I_2 R}{R_{eq}}$.

∴ from $I_1 V_1 = I_2 V_2$

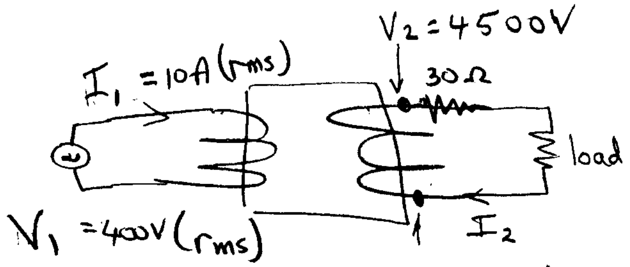
$$\frac{N_1}{N_2}\frac{\cancel{I_2} R}{R_{eq}} \cdot \frac{N_1}{N_2}\cancel{V_2} = \cancel{I_2}\cancel{V_2}$$

$$\Rightarrow R_{eq} = \left(\frac{N_1}{N_2}\right)^2 R$$





Example: A Step-Up Transformer



find % power lost.

$$I_2 = \frac{I_1 V_1}{V_2} = \frac{10 \times 400}{4500} = 0.89\,A$$

$\therefore$ Power in R on secondary side $= P_{lost} = I_2^2 R$

$$= (.89)^2(30)$$
$$= 24\,W$$

Total power in $= I_1 V_1 = 4000$ W.

$\therefore$ % lost $= \frac{24}{4000} \times 100\% = 0.60\%$

If a transformer had not been used:

$I_2 = I_1 = 10$ A

$\therefore P_{lost} = I_2^2 R = 3000$ W.

$\therefore$ % lost $= \frac{3000}{4000} \times 100\% = 75\%$

If transmission line is cooled so that $30 \to 5\,\Omega$, what is power lost if $I_2 = 0.89$ A?

$$P_{lost} = (0.89)^2(5) = 4.0\ W,$$

showing the advantage of cooling. Cooling is costly, however.