# Neutron Transport Equation

Prepared by
Dr. Daniel A. Meneley, Senior Advisor, Atomic Energy of Canada Ltd. and
Adjunct Professor, Department of Engineering Physics
McMaster University, Hamilton, Ontario, Canada

**Summary:**

Derivation of the low-density Boltzmann equation for neutron transport, from the first principles. Examination of the approximations inherent in the formulation.

More about this document

## Table of Contents

We now set out to calculate the rate of fission at all locations inside a fission reactor. To do this we must first solve for the space-energy-time distribution of the neutrons that cause fission.

Imagine yourself as a neutral particle smaller than one pico-metre in diameter – a neutron. (You might prefer to think of yourself as a wave. This is okay.) What do you "see" as you travel? The view might be similar to what humans see when they look into a night sky – mostly space. The stuff that humans see as solid is, in fact, mostly space. Atoms are encountered rarely as you travel through this space. Many electrons surround these atoms but, because you have zero charge, you are not attracted to them. You see several different kinds of nuclei as you travel through space.

Occasionally, you (the neutron) "bounce off" a nucleus and fly off in a new direction. Each time you bounce you lose some of your energy. As your energy becomes lower the nuclei around you appear to be somewhat larger and you become more likely to hit them. At certain neutron energies, some nuclei are very "large" and you collide with them quite easily. The results of your collision depend on the distance between your direction vector and the nucleus center of mass. And sometimes, when you collide, you are "captured" (absorbed). If you are unlucky you fly out of the reactor altogether into a nearly empty space where you travel long distances until you are either captured or until you decay into a proton and electron, and lose your identity.

You might be fortunate enough to reach a warm place where some collisions actually increase your energy – you are now "thermalized". You scatter along happily until, suddenly, you are captured by a nucleus.

If you are lucky you are captured by a uranium 235 nucleus and cause it to break apart. You then will be reincarnated along with two or more close neutron relatives, and will begin your journey again. In most cases you exist in a cloud of many trillions of almost identical relatives (differing only in their energy and direction) but you hardly ever meet one.

This fanciful tale might help you to visualize our mathematical challenge. We must solve an equation that describes the space, energy, and time distribution of neutrons in heterogeneous media. The density of neutrons is so very high that we need to calculate only their ensemble average behaviour to solve for the local fission rate. At the same time the density of neutrons is low enough that we need not consider neutron-neutron reactions. Densities of nuclei in the media are low so that neutron-nucleus interactions can be considered, in most cases, to be singular (i.e. they occur one at a time). Exceptions to this rule occur in strongly absorbing situations such as at energies near a strong nuclear resonance. We must then take account of "self-shielding". Some exceptions occur at low neutron energies, such as in water, where molecular binding effects are important.[Your words imply that self-shielding is associated with non-singular events. In a resonance, the events still occur one at a time do they not? But even if they don't, you can have self-shielding effects without a resonance. Self-shielding is more associated with flux dips due to absorption that you can't easily capture with an 'average' flux over the region. I always associated the term with modelling limitations, not with any unique phenomena. A current of neutrons is always self-shielded in the sense that a detector perturbs the flux that it is trying to measure.]

Methods used to approximate the actual neutron-nucleus reaction probabilities are presented in many well-formulated textbooks. In the remainder of this course we will assume that in any reaction, we know the probability of interaction of a neutron with a nucleus for any given neutron energy and "collision angle".

**NEUTRON BALANCE EQUATION**

The neutron density n = n ($\underline{r}$, E, $\underline{O}$, t) is, in general, a function of spatial position $\underline{r}$, energy E, angle $\underline{\Omega}$ and time t. It exists, in general, in a heterogeneous reactor environment where the material properties also are a function of $\underline{r}$, E, $\underline{O}$, and t. We treat the neutron population as a continuum, avoiding statistical fluctuations. Typically, $f \equiv nv \sim 10^{14}$ neutrons/cm$^2$-sec. Therefore, the statistical fluctuations are negligible. Also we ignore neutron-neutron interactions since the neutron density is small compared to the density of the medium (~ $10^{22}$ atoms/cm$^3$.).

The continuity or conservation equation, based on our intuitive experience, states:

$$\frac{d}{dt}\int_{\cancel{V}} n\,d\cancel{V} = \int_{\cancel{V}}\left(\sum_i S_i\right)d\cancel{V} \qquad \text{EQ. 1}$$

where $S_i$ represents any neutron source or sink.

| | | |
|---|---|---|
| *the substantial derivative of the neutron population in a volume,* $\cancel{V}$ | $\}$ = | *sum of sinks and sources in that volume* |

The Reynold's Transport Theorem states:

$$\frac{d}{dt}\int_{\cancel{V}} \Psi\,d\cancel{V} = \int_{\cancel{V}}\frac{\partial\Psi}{\partial t}\,d\cancel{V} + \int_{S}\Psi\underline{v}\cdot d\underline{S} \qquad \text{EQ. 2}$$

(total) = (generation) + (outflow)

where $\Psi$ is any field parameter,

$\underline{v}$ = velocity of the parameter ($\underline{v}$ = v $\underline{O}$)

$\underline{S}$ = normal surface vector, and

$\cancel{V}$ = volume

thus:

$$\int_{\cancel{V}} \frac{\partial n}{\partial t}\, d\cancel{V} \;=\; -\int_{S} n\underline{v}\cdot d\underline{S} \;+\; \int_{\cancel{V}} \left( \sum_{i} S_{\boldsymbol{i}} \right) d\cancel{V} \qquad \text{EQ. 3}$$

Using Gauss' divergence theorem:

$$\int_{S} \underline{A}\cdot d\underline{S} \;=\; \int_{\cancel{V}} \nabla \cdot A\, d\cancel{V} \qquad \text{EQ. 4}$$

gives:

$$\int_{\cancel{V}} \frac{\partial n}{\partial t}\, d\cancel{V} \;=\; -\int_{\cancel{V}} \nabla\cdot(n\underline{v})\; d\cancel{V} \;+\; \int_{\cancel{V}} \sum_{i} S_{\boldsymbol{i}}\; d\cancel{V} \qquad \text{EQ. 5}$$

or:

$$\frac{\partial\, n}{\partial\, t} = \;\; -\nabla\cdot n\underline{v} + \sum_{i} S_{\boldsymbol{i}} \qquad \text{EQ. 6}$$

where: $n = n\,(\underline{r}, E, \underline{O}, t)$ *(neutron density)*

$v = v\,(E)$ *(neutron velocity corresponding to energy,* $E$ *)*

$\underline{v} = v\,\underline{O}$ *(velocity of the parameter)*

$S_{\boldsymbol{i}} = S_{\boldsymbol{i}}\,(\underline{r}, E, \underline{O}, t)$ *(neutron source or sink density)*

We further limit ourselves to the following sinks and sources:

1. prompt fission neutron source
2. scattering into ($\underline{r}$, E, $\underline{O}$, t).
3. delayed neutron precursor source
4. loss out of ($\underline{r}$, E, $\underline{O}$, t) by absorption or scattering.
5. external neutron sources.

**Prompt Fission Source**

To create n neutrons at some $\underline{r}$, E, $\underline{O}$, t from n' neutrons at $\underline{r}'$, E', $\underline{O}'$, t', the n' neutrons are absorbed at the ' conditions, causing fission in which neutrons are created. The process is a function of both the primed and unprimed conditions.

Hence:

$$S_{FISSION} = \int_{E'}\int_{\underline{O}} v' n' v' (1-b')\; \Sigma_{f'}\, ?_{p}\; d\underline{\Omega}'\;\; dE' \qquad \text{EQ. 7}$$

where: $v' = v(E')$
$= $ *neutron velocity at energy E'*

$n' = n(\underline{r}, E', \underline{O}', t)$
$= $ *neutron density*

$\nu' = \nu(\underline{r}, E', \underline{\Omega}', t)$
$= $ *total number of fission neutrons produced per fission, in the fuel, caused by a neutron of energy* $E'$*, direction* $\underline{O}'$.

$b' = b(\underline{r}, E', \underline{\Omega}', t)$
$= \sum_i b_i$
$= $ *delayed neutron fraction (fraction of all fission neutrons that are delayed in time, following fission)*

$\Sigma_{f'} = \Sigma_{f'}(\underline{r}, E', \underline{\Omega}', t)$
$= $ *fission cross section.*

$?_p = ?_p(\underline{r}, E, \underline{\Omega}, t)$
$= $ *prompt neutron energy emission spectrum.*

By integrating over E' and O', we get the total fission contribution of fission neutrons ending up in the interval about E and O.

Things to note:

1) r' = r because a fission creates neutrons "on the spot".

2) t' = t because the time delay is very small (~$10^{-14}$ sec).

3) The product: $\int_{E'}\int_{\underline{O}} v' n' \nu' (1-b)\ \Sigma_{f'} d\underline{\Omega}'\ dE'$
gives the total number of fission neutrons produced per second. The spectrum factor, $?_p$, proportions this total to the respective energy and direction interval dE, dO about E, $\underline{O}$.

4) The units of n ($\underline{r}$, E, $\underline{O}$, t) are neutrons per unit space per unit energy, per unit time i.e. #/($cm^3$-eV - steradian-sec).

**Scattering Source**

Neutrons at E', $\underline{O}'$ are scattered out of E', $\underline{O}'$ into E, $\underline{O}$ (really the interval dE' d$\underline{O}'$, about E', $\underline{O}'$, etc.), according to the probability density function.

$f_s(E', \underline{\Omega}', t \rightarrow E', \underline{\Omega}', t)$ and the macroscopic scattering cross section $\Sigma_s(\underline{r}, E', \underline{\Omega}', t)$

Thus:

$$S_{SCATTER-IN} = \int_{E'}\int_{\underline{\Omega}} v(E')\ \underline{\Omega'} n\left(\underline{r}, E', \underline{\Omega'}, t\right)\ \Sigma_S\left(\underline{r}, E', \underline{\Omega'}, t\right) f_s\left(\underline{r}, E', \underline{\Omega'} \rightarrow E, \underline{\Omega}, t\right) dE'\, d\underline{\Omega'} \qquad \text{EQ. 8}$$

**Delayed Neutron Source**

The delayed neutron source term arises by decay from several precursor isotopes, denoted as $C_i$ (r, t). Each precursor isotope ($i$) emits a spectrum of neutron energies: $?_i$ (E, O).

Thus:

$$S_{PRECURSOR} = \sum_{i=1}^{N} l_i\ ?_i\ (E, \underline{\Omega})\ C_i(\underline{r}, t) \qquad \text{EQ. 9}$$

In rare circumstances (such as calculations of energetic disassembly of fast reactors, or fluid-fuelled reactors), the geometric frame of reference of delayed neutron precursors moves relative to that of the neutron fluxes. Reference IV used a combination of Euler and Lagrange (mass conservative) coordinate systems to track these relative movements.

**Loss Terms**

Loss by absorption and scattering follow directly as:

$$S_{ABSORPTION} = -v\ (E)\ n(\underline{r}, E, \underline{\Omega}, t)\ \Sigma_a(\underline{r}, E, \underline{\Omega}, t) \qquad \text{EQ. 10}$$

$$S_{SCATTEROUT} = -v\ (E)\ n(\underline{r}, E, \underline{\Omega}, t)\ \Sigma_S(\underline{r}, E, \underline{\Omega}, t) \qquad \text{EQ. 11}$$

Loss by transport completely out of the reactor is a special case of EQ. 8, where the number of neutrons returning across the outer surface is taken to be zero.

**External Sources**

In some cases we must include neutrons that are produced in the reactor but independent of the fission chain. When the reactor is at high power such sources are negligible compared with the large number of neutrons produced in fission. But there are relatively rare, yet crucially important, situations in which the magnitude and distribution of external sources can have a dramatic influence on the safety of the reactor. Because these situations are rare they tend to be neglected. In this course we make a deliberate attempt to correct this deficiency.

$$S_{EXTERNAL} = S(\underline{r}, E, \underline{\Omega}, t) \qquad \text{EQ. 12}$$

SUMMING UP

Summing these up and plugging into EQ. 6 gives the **Boltzmann Transport Equation**:

$$\frac{\partial n}{\partial t} = \int_{E'}\int_{\underline{\Omega}} v'n' \left\{ \nu'(1-ß')\Sigma_{f'} \, ?_p + \Sigma_s \, f_s \right\} d\underline{\Omega}' \, dE' - \nabla \cdot n\underline{v}$$
$$+ \sum_{i=1}^{N} l_i \, ?_i C_i - vn\,(\Sigma_a + \Sigma_s) + S_{EXTERNAL} \qquad \text{EQ. 13}$$

Similarly, the delayed neutron precursor concentration equation is:

$$\left(\frac{\partial C_i}{\partial t}\right) = -l_i C_i + \int_{E'}\int_{\underline{O}} \left( b_i' \, \nu' v' n' \, \Sigma_{f'} \right) d\underline{\Omega}' \, dE' \qquad \text{EQ. 14}$$

Although these equations are very general and difficult to solve directly, they embody several assumptions, as already noted. We'll make a few more approximations to get to the stage where we can solve these equations on a routine basis.

Note that the capability of modern computers has made a dramatic difference to reactor physics. Over the past four decades we have progressed to the point that many approximations that were essential in the 1940-1960 time frame can now simply be discarded. However, some of these approximations are still used, both to improve the economics of analysis and to permit easier understanding of results. Detailed models produce a vast array of numerical data. Summaries and approximations enhance our understanding of these data.

Assumptions so far:

1) t' = t
2) r' = r
3) no statistical fluctuations
4) no sinks or sources other than those listed
5) no neutron - neutron interactions

THINGS TO CONTEMPLATE

1) Can you find any other assumptions?
2) This continuity equation is similar to the conservation of mass in thermalhydraulics. Why do we not also generally consider the conservation of momentum and energy? *
3) Expand on the implications of t' = t, r' = r.

## References

I. See Appendix 1, Section 1.1 (Handout).
II. See Appendix 2, Reynold's Transport Theorem.
III. See any of the classics, like Bell & Glasstone.
IV. W.T. Sha, et. al.: "Two-Dimensional Fast-Reactor Disassembly Analysis with Space-Time Kinetics", CONF-710302 (Vol. 1) Proc. Conf. on New Developments in Reactor Mathematics and Applications, Idaho (March 1971).

Back to page 1

About this document

Author and affiliation: Daniel A. Meneley, Adjunct Professor, Department of Engineering Physics, McMaster University, Ontario, Canada; Senior Advisor, Marketing and Sales, Atomic Energy of Canada Ltd., Mississauga, Ontario, Canada.

Revision history:

- Revision 0, February 16, 2001, draft 1
- Source document archive location: h:\Violeta 1 – Word\web\Neutron Transport Equation.doc
- Contact person: Violeta Sibana